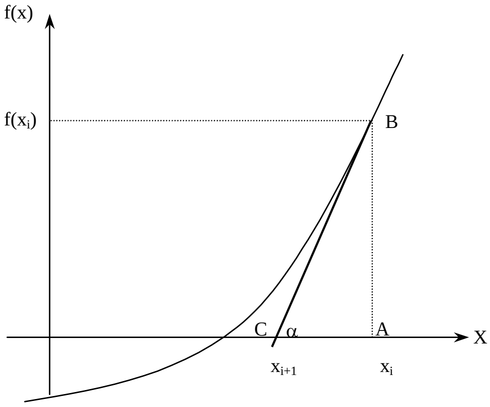
Newton-Raphson Method
to solve a nonlinear equation *f* (*x*)=0



$$\tan(\alpha) = \frac{AB}{AC}$$

$$f'(x_i) = \frac{f(x_i)}{x_i - x_{i+1}}$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

System of nonlinear equations

$$f_1(x, y) = x^2 - 2x - y + 0.5$$
$$f_2(x, y) = x^2 + 4y^2 - 4.$$

$$f_1(x, y) = 0 \quad \text{and} \quad f_2(x, y) = 0.$$

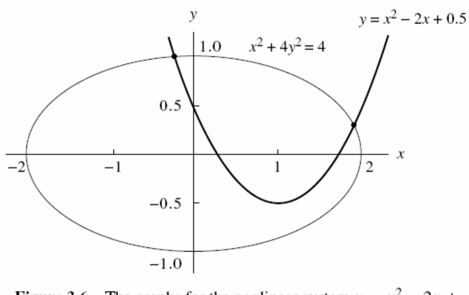


Figure 3.6 The graphs for the nonlinear system $y = x^2 - 2x + 0.5$ and $x^2 + 4y^2 = 4$.

Let $$u = f_1(x, y)$$ $$v = f_2(x, y)$$

And then consider a system of linear approximations near (x_0, y_0)

$$u - u_0 \approx \frac{\partial}{\partial x} f_1(x_0, y_0)(x - x_0) + \frac{\partial}{\partial y} f_1(x_0, y_0)(y - y_0),$$
$$v - v_0 \approx \frac{\partial}{\partial x} f_2(x_0, y_0)(x - x_0) + \frac{\partial}{\partial y} f_2(x_0, y_0)(y - y_0).$$

$$\begin{bmatrix} u - u_0 \\ v - v_0 \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} f_1(x_0, y_0) & \frac{\partial}{\partial y} f_1(x_0, y_0) \\ \frac{\partial}{\partial x} f_2(x_0, y_0) & \frac{\partial}{\partial y} f_2(x_0, y_0) \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}$$

$$\Uparrow$$

$$\Delta \boldsymbol{F} \approx \boldsymbol{J}(x_0, y_0)\, \Delta \boldsymbol{X}$$ (***J*** is called the Jacobian)

u=0, *v*=0 is equivalent to $$0 = f_1(x, y)$$ $$0 = f_2(x, y)$$

If (*p*,*q*) is a solution, then $$0 = f_1(p, q)$$ $$0 = f_2(p, q).$$

For small changes near (p_0,q_0) $$\Delta p = x - p_0.$$ $$\Delta q = y - q_0.$$

substituting $$u - u_0 = f_1(p, q) - f_1(p_0, q_0) = 0 - f_1(p_0, q_0)$$ $$v - v_0 = f_2(p, q) - f_2(p_0, q_0) = 0 - f_2(p_0, q_0)$$

in $$\begin{bmatrix} u - u_0 \\ v - v_0 \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} f_1(x_0, y_0) & \frac{\partial}{\partial y} f_1(x_0, y_0) \\ \frac{\partial}{\partial x} f_2(x_0, y_0) & \frac{\partial}{\partial y} f_2(x_0, y_0) \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}$$

we get $$\begin{bmatrix} \frac{\partial}{\partial x} f_1(p_0, q_0) & \frac{\partial}{\partial y} f_1(p_0, q_0) \\ \frac{\partial}{\partial x} f_2(p_0, q_0) & \frac{\partial}{\partial y} f_2(p_0, q_0) \end{bmatrix} \begin{bmatrix} \Delta p \\ \Delta q \end{bmatrix} \approx - \begin{bmatrix} f_1(p_0, q_0) \\ f_2(p_0, q_0) \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial}{\partial x} f_1(p_0, q_0) & \frac{\partial}{\partial y} f_1(p_0, q_0) \\ \frac{\partial}{\partial x} f_2(p_0, q_0) & \frac{\partial}{\partial y} f_2(p_0, q_0) \end{bmatrix} \begin{bmatrix} \Delta p \\ \Delta q \end{bmatrix} \approx - \begin{bmatrix} f_1(p_0, q_0) \\ f_2(p_0, q_0) \end{bmatrix}$$

$$\Delta \boldsymbol{P} \approx -\boldsymbol{J}(p_0, q_0)^{-1} \boldsymbol{F}(p_0, q_0)$$

$$\Delta \boldsymbol{P} = \begin{bmatrix} \Delta p & \Delta q \end{bmatrix}'$$

$$\boldsymbol{P}_1 = \boldsymbol{P}_0 + \Delta \boldsymbol{P} = \boldsymbol{P}_0 - \boldsymbol{J}(p_0, q_0)^{-1} \boldsymbol{F}(p_0, q_0).$$

Compare to $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$

Step 1. Evaluate the function

$$\boldsymbol{F}(\boldsymbol{P}_k) = \begin{bmatrix} f_1(p_k, q_k) \\ f_2(p_k, q_k) \end{bmatrix}.$$

Step 2. Evaluate the Jacobian

$$\boldsymbol{J}(\boldsymbol{P}_k) = \begin{bmatrix} \frac{\partial}{\partial x} f_1(p_k, q_k) & \frac{\partial}{\partial y} f_1(p_k, q_k) \\ \frac{\partial}{\partial x} f_2(p_k, q_k) & \frac{\partial}{\partial y} f_2(p_k, q_k) \end{bmatrix}.$$

Step 3. Solve the linear system

$$\boldsymbol{J}(\boldsymbol{P}_k)\Delta \boldsymbol{P} = -\boldsymbol{F}(\boldsymbol{P}_k) \quad \text{for } \Delta \boldsymbol{P}.$$

Step 4. Compute the next point:

$$\boldsymbol{P}_{k+1} = \boldsymbol{P}_k + \Delta \boldsymbol{P}.$$

Now, repeat the process.

Example 3.32

$$0 = x^2 - 2x - y + 0.5$$
$$0 = x^2 + 4y^2 - 4.$$

$$F(x, y) = \begin{bmatrix} x^2 - 2x - y + 0.5 \\ x^2 + 4y^2 - 4 \end{bmatrix}, \quad J(x, y) = \begin{bmatrix} 2x - 2 & -1 \\ 2x & 8y \end{bmatrix}$$

$(p_0, q_0) = (2.00, 0.25)$ $\quad F(2.00, 0.25) = \begin{bmatrix} 0.25 \\ 0.25 \end{bmatrix}, \quad J(2.00, 0.25) = \begin{bmatrix} 2.0 & -1.0 \\ 4.0 & 2.0 \end{bmatrix}$

$J(P_k)\Delta P = -F(P_k)$ $\quad \begin{bmatrix} 2.0 & -1.0 \\ 4.0 & 2.0 \end{bmatrix} \begin{bmatrix} \Delta p \\ \Delta q \end{bmatrix} = -\begin{bmatrix} 0.25 \\ 0.25 \end{bmatrix} \quad \begin{bmatrix} \Delta p \\ \Delta q \end{bmatrix} = \begin{bmatrix} -0.09375 \\ 0.0625 \end{bmatrix}$

$$P_1 = P_0 + \Delta P = \begin{bmatrix} 2.00 \\ 0.25 \end{bmatrix} + \begin{bmatrix} -0.09375 \\ 0.0625 \end{bmatrix} = \begin{bmatrix} 1.90625 \\ 0.3125 \end{bmatrix}$$

Example 3.32

$$0 = x^2 - 2x - y + 0.5$$
$$0 = x^2 + 4y^2 - 4.$$

Table 3.7 Function Values, Jacobian Matrices, and Differentials Required for Each Iteration in Newton's Solution to Example 3.32

P_k	Solution of the linear system $J(P_k)\Delta P = -F(P_k)$	$P_k + \Delta P$
$\begin{bmatrix} 2.00 \\ 0.25 \end{bmatrix}$	$\begin{bmatrix} 2.0 & -1.0 \\ 4.0 & 2.0 \end{bmatrix} \begin{bmatrix} -0.09375 \\ 0.0625 \end{bmatrix} = -\begin{bmatrix} 0.25 \\ 0.25 \end{bmatrix}$	$\begin{bmatrix} 1.90625 \\ 0.3125 \end{bmatrix}$
$\begin{bmatrix} 1.90625 \\ 0.3125 \end{bmatrix}$	$\begin{bmatrix} 1.8125 & -1.0 \\ 3.8125 & 2.5 \end{bmatrix} \begin{bmatrix} -0.005559 \\ -0.001287 \end{bmatrix} = -\begin{bmatrix} 0.008789 \\ 0.024414 \end{bmatrix}$	$\begin{bmatrix} 1.900691 \\ 0.311213 \end{bmatrix}$
$\begin{bmatrix} 1.900691 \\ 0.311213 \end{bmatrix}$	$\begin{bmatrix} 1.801381 & -1.000000 \\ 3.801381 & 2.489700 \end{bmatrix} \begin{bmatrix} -0.000014 \\ 0.000006 \end{bmatrix} = -\begin{bmatrix} 0.000031 \\ 0.000038 \end{bmatrix}$	$\begin{bmatrix} 1.900677 \\ 0.311219 \end{bmatrix}$

Error definition for stopping criteria

Definition 3.7. Suppose that X and Y are two points in N-dimensional space. We define the distance between X and Y in the $\|*\|_1$ norm as

$$\|X - Y\|_1 = \sum_{j=1}^{N} |x_j - y_j|. \qquad \blacktriangle$$

$$\|X\|_1 = \sum_{j=1}^{N} |x_j|.$$

$P = (2, 4, 3)$ and $Q = (1.75, 3.75, 2.95)$.

$$\|P - Q\|_1 = |2 - 1.75| + |4 - 3.75| + |3 - 2.95| = 0.55.$$

$$error = \|P_{k+1} - P_k\|_1 \qquad relative\ error = \frac{\|P_{k+1} - P_k\|_1}{\|P_k\|_1}$$

Step 1. Evaluate the function

$$\boldsymbol{F}(\boldsymbol{P}_k) = \begin{bmatrix} f_1(p_k, q_k) \\ f_2(p_k, q_k) \end{bmatrix}.$$

Step 2. Evaluate the Jacobian

$$\boldsymbol{J}(\boldsymbol{P}_k) = \begin{bmatrix} \frac{\partial}{\partial x} f_1(p_k, q_k) & \frac{\partial}{\partial y} f_1(p_k, q_k) \\ \frac{\partial}{\partial x} f_2(p_k, q_k) & \frac{\partial}{\partial y} f_2(p_k, q_k) \end{bmatrix}.$$

Step 3. Solve the linear system

$$\boldsymbol{J}(\boldsymbol{P}_k)\Delta \boldsymbol{P} = -\boldsymbol{F}(\boldsymbol{P}_k) \qquad \text{for } \Delta \boldsymbol{P}.$$

Step 4. Compute the next point:

$$\boldsymbol{P}_{k+1} = \boldsymbol{P}_k + \Delta \boldsymbol{P}.$$

Now, repeat the process.

$$F = \begin{matrix} f_1(x, y, z) \\ f_2(x, y, z) \\ f_3(x, y, z) \end{matrix}$$

$$J(x, y, z) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{bmatrix}.$$

$$J(P_k)\Delta P = -F(P_k)$$

$$P_{k+1} = P_k + \Delta P.$$