

# Uniform Asymptotic Expansions of Multiple Scattering Iterations

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## Abstract

Although every implementation of a recent high frequency multiple scattering solver has displayed a frequency independent operation count, its numerical analysis yet remains as a challenging open problem. This is, in part, due to the absence of detailed information on the uniform asymptotic expansions of multiple scattering iterations. Here we address precisely this issue for a collection of convex obstacles in both two and three space dimensions and further, as an application, we present a generalized geometrical optics solver.

## Introduction

Although every implementation of a recent high frequency multiple scattering solver [1] has displayed a frequency independent operation count to attain a prescribed accuracy, its numerical analysis yet remains as a challenging open problem. This is, in part, due to the absence of detailed information on the uniform asymptotic expansions of multiple scattering iterations. Indeed, asymptotic expansions, in their full generality, are only known for a *single* convex obstacle illuminated by a plane-wave incidence [2] and this, in turn, has given rise to the development of asymptotically  $\mathcal{O}(1)$  single-scattering solvers [1], [3], [4]. Here we extend the results in [2] to encompass a collection of compact strictly convex obstacles and thereby enable a straightforward extension of the single-scattering algorithms [3], [4] to accompany the multiple scattering solver in [1]. Further, as an application of our derivations, we present a generalized geometrical optics solver.

## 1 Multiple scattering

We consider here the sound soft acoustic scattering problem [5] from a smooth compact obstacle  $K$  in  $\mathbb{R}^n$ ,  $n = 2, 3$ , whose solution can be expressed as a single-layer potential with unknown density  $\eta$ , the normal derivative of the total field on  $\partial K$ . Although a variety of integral equations exist for  $\eta$ , for simplic-

ity, we use here

$$\eta(x) - \int_{\partial K} \frac{\partial G(x, y)}{\partial \nu(x)} \eta(y) ds(y) = 2 \frac{\partial u^{\text{inc}}(x)}{\partial \nu(x)}, \quad x \in \partial K$$

where  $G = -2\Phi$  and  $\Phi$  is the outgoing fundamental solution to the Helmholtz equation. As is well-known [1], [5], when the obstacle  $K$  consists of finitely many connect components  $\{K_\sigma : \sigma \in \mathcal{I}\}$ , writing the preceding equation in component form and inverting the diagonal part gives rise to an alternative (operator) equation whose Neumann series solution corresponds precisely to multiple scattering; and this, in turn, can be shown to imply that  $\eta$  is the superposition over all obstacle paths  $\{K_m\}_{m \geq 0} \subset \{K_\sigma : \sigma \in \mathcal{I}\}$  with  $K_{m+1} \neq K_m$  (rearranged suitably) of the multiple scattering iterations  $\eta_m$  that recursively solve on  $\partial K_m$  the integral equations

$$\eta_0(x) - \int_{\partial K_0} \frac{\partial G(x, y)}{\partial \nu(x)} \eta_0(y) ds(y) = 2 \frac{\partial u^{\text{inc}}(x)}{\partial \nu(x)}$$

and, for  $m \geq 1$ ,

$$\begin{aligned} \eta_m(x) - \int_{\partial K_m} \frac{\partial G(x, y)}{\partial \nu(x)} \eta_m(y) ds(y) \\ = \int_{\partial K_{m-1}} \frac{\partial G(x, y)}{\partial \nu(x)} \eta_{m-1}(y) ds(y). \end{aligned}$$

As is further known, when the obstacles  $K_\sigma$  are strictly convex and satisfy the *visibility* and *no-occlusion* conditions (cf. [5]), the multiple scattering iterations  $\eta_m$  admit the factorizations

$$\eta_m(x) = e^{ik\varphi_m(x)} \eta_m^{\text{slow}}(x), \quad x \in \partial K_m. \quad (1)$$

Here the phase functions  $\varphi_m$  are defined on  $\partial K_m$  as

$$\varphi_m(x) = \begin{cases} \alpha \cdot x, & m = 0 \\ |x - \mathcal{X}_{m-1}^m(x)| + \varphi_{m-1}(\mathcal{X}_{m-1}^m(x)), & m \geq 1 \end{cases}$$

where  $(\mathcal{X}_0^m(x), \dots, \mathcal{X}_m^m(x)) \in \partial K_0 \times \dots \times \partial K_m$  are the (uniquely determined) reflection points that, at each reflection, specify the geometrical *illuminated regions*  $\partial K_m^{IL}$ , *shadow regions*  $\partial K_m^{SR}$ , and *shadow boundaries*  $\partial K_m^{SB}$ .

## 2 Asymptotic expansions

The uniform asymptotic expansions of the slow densities  $\eta_m^{\text{slow}}$  are now summarized in the following.

**Theorem (i)**  $\eta_m^{\text{slow}} \in S_{1,0}^1(\partial K_m^{IL} \times (0, \infty))$  (see [5], [2] for the definition of  $S_{\rho,\sigma}^\mu$ ) and

$$\eta_m^{\text{slow}}(x, k) \sim \sum_{j \geq 0} k^{1-j} a_{m,j}(x)$$

where  $a_{m,j}(x)$  are complex-valued  $C^\infty$  functions.

**(ii)**  $\eta_m^{\text{slow}} \in S_{2/3,1/3}^1(\partial K_m \times (0, \infty))$  and

$$\eta_m^{\text{slow}}(x, k) \sim \sum_{p,q \geq 0} k^{2/3-2p/3-q} b_{m,p,q}(x) \Psi^{(p)}(k^{1/3} Z_m(x))$$

where  $b_{m,p,q}(x)$  are complex-valued  $C^\infty$  functions,  $Z_m(x)$  is a real-valued  $C^\infty$  function that is positive on  $\partial K_m^{IL}$ , negative on  $\partial K_m^{SR}$ , and vanishes precisely to first order on  $\partial K_m^{SB}$ . The function  $\Psi$ , on the other hand, is as specified in [2].

## 3 Numerical example

As an application of this theorem, we present a comparison of the multiple scattering iterations  $\eta_m$  and the *generalized geometrical optics approximations*  $\eta_m^{GO}$  defined as

$$\eta_m^{GO}(x) = e^{ik\phi_m(x)} k \begin{cases} a_{m,0}(x), & x \in \partial K_m^{IL}, \\ 0, & \text{otherwise} \end{cases}$$

(see [5] for the description of  $a_{m,0}$ ). Specifically, Figure 1 displays the plots of the maximum of  $\mu_m(x) = \log_{10}[|\eta_m(x) - \eta_m^{GO}(x)|/k]$  against  $\log_{10} k$  (a) on a fixed compact subset  $\mathcal{S}$  of  $\partial K_m^{IL}$  justifying the first part of the Theorem (the middle sub-figure), and (b) over the entire boundary  $\partial K_m$  verifying the second part of the Theorem (the bottom sub-figure) for the given two-ellipse configuration on the periodic orbit  $K_0, K_1$  for the specific value  $m = 8$ .

## References

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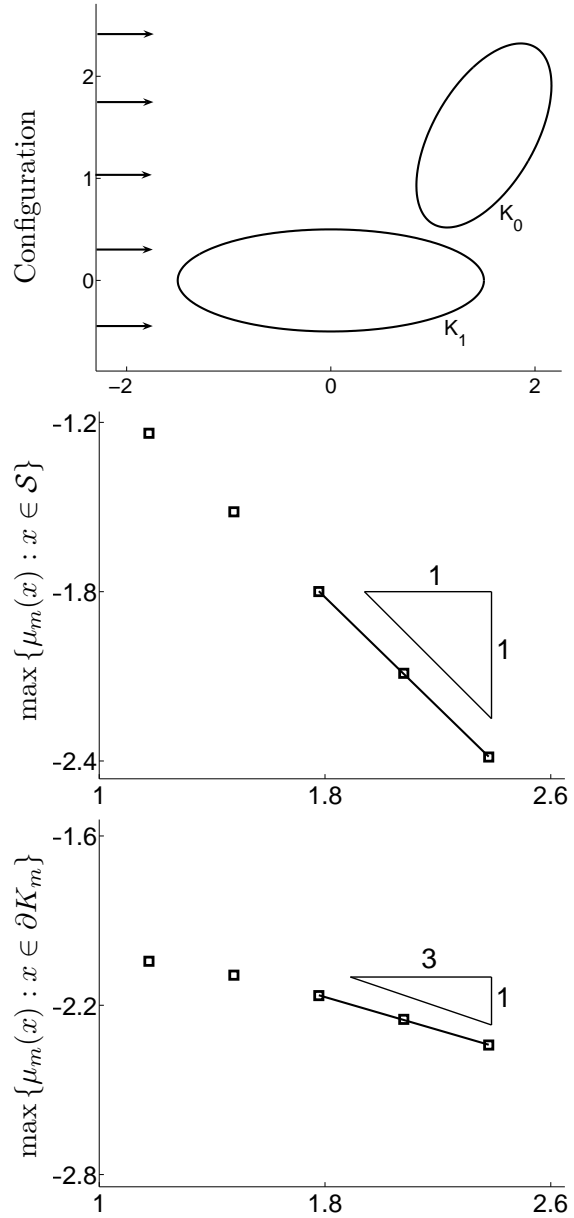


Figure 1: A two-ellipse configuration illuminated by a plane-wave incidence  $e^{ik\alpha \cdot x}$ ,  $\alpha = (0, 1)$ .

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