Decay of Multiple-scattering Iterates for Trapping Obstacles in the High-frequency Regime

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Introduction

The numerical evaluation of electromagnetic/acoustic scattering returns from a bounded obstacle that admits only a finite number of ray reflections within itself, can be performed in frequency-independent computational times within prescribed error-tolerances [2]. On the other hand, for obstacles consisting of many connected components, and that therefore give rise to trapped rays, the solution of the scattering problem can be represented as an infinite multiple-scattering series, and this representation can be used to generalize the numerical algorithm of [2] to more general configurations [1]. In this paper, we present a novel analysis of this multiple scattering series in the high-frequency regime, and we obtain analytic formulas for its rate of decay.

Formulation, Integral Equations and Multiple-scattering

The classical obstacle-scattering problem consists of evaluating the scattering of an incident plane wave $u^{inc}(x) = e^{ik\alpha\cdot x}$, $|\alpha| = 1$, from a bounded obstacle K. For simplicity of our presentation, we restrict ourselves to the two-dimensional context and consider the Dirichlet boundary conditions (TE polarization in electromagnetics). In which case, the relevant frequency-domain scattering-problem is

$$\begin{aligned} \Delta u(x) + k^2 u(x) = 0, &\quad x \in \mathbf{R}^2 \backslash \overline{K}, \\ u(x) = -u^{inc}(x), &\quad x \in \partial K. \end{aligned} \tag{1}$$

An integral equation formulation of (1) is given by

$$\eta - R\eta = 2\partial u^{inc}/\partial\nu, \qquad \text{on } \partial K. \tag{2}$$

Here ν is the outward unit normal to ∂K, $\eta = \partial u/\partial\nu$ is the *surface current*, and

$$R\eta(x) = -2\int_{\partial K} \frac{\partial\Phi(x,y)}{\partial\nu(x)}\eta(y)ds(y)$$

where Φ is the radiating free-space Green function. When $K = \cup\{K_\sigma : \sigma \in I, |I| < \infty\}$ is a finite union of disjoint sets, equation (2) takes on the form

$$\eta_\sigma - R_{\sigma\sigma}\eta_\sigma - \sum_{\rho\in I\setminus\{\sigma\}} R_{\sigma\rho}\eta_\rho = f_\sigma, \qquad \sigma \in I \tag{3}$$

where $\eta_\sigma = \eta|_{\partial K_\sigma}$, $f_\sigma = (2\partial u^{inc}/\partial\nu)|_{\partial K_\sigma}$, and on ∂K_σ

$$R_{\sigma\rho}\eta_\rho(x) = -2\int_{\partial K_\rho} \frac{\partial\Phi(x,y)}{\partial\nu(x)}\eta_\rho(y)ds(y), \qquad \rho \in I.$$

As is apparent from (3) then, the total surface current η, being the solution of the operator equation of the second kind (2), is the superposition of the solutions of the integral equations

$$\eta_0 - R_{0,0}\eta_0 = f_0 \qquad \text{on } \partial K_0 \quad m = 0 \tag{4}$$

$$\eta_m - R_{m,m}\eta_m = R_{m,m-1}\eta_{m-1} \qquad \text{on } \partial K_m \quad m = 1,2,\dots \tag{5}$$

over all obstacle paths $\{K_m\}_{m\geq 0} \subset \{K_\sigma : \sigma \in I\}$ where no two consecutive objects are the same. The significance of this formulation stems from the fact that it guarantees that each of the problems in (4)−(5) entails the solution of problems within single scattering configurations for which the methods described in [1, 2] provide an error-controllable scheme with fixed computational complexity.

High-frequency Asymptotic Expansions of Multiple-scattering Iterates

Suppose that the obstacles $\{K_\sigma : \sigma \in I\}$ are convex, and are *visible* in the sense that no K_σ meets with the convex hull of any other pair of obstacles. For a fixed m, and a fixed $x_m \in \partial K_m$, $(x_0,\dots,x_{m-1}) \in \partial K_0 \times \cdots \times \partial K_{m-1}$ will denote the unique set of points determined by the geometrical optics solution. We also set $\nu_m := \nu(x_m)$, and $\varphi_m = \varphi_m(x_m) = \alpha\cdot x_0 + \sum_{j=0}^{m-1}|x_{j+1}-x_j|$. Our first main result states that, in the high-frequency regime, the behavior of the currents, that is the solutions η_0, η_1, …, η_m, … of (4)−(5), depend solely on the geometrical characteristics of the surfaces ∂K_j on the optical ray paths.

Theorem [3] *For $m = 0,1,\dots$, the high-frequency asymptotic expansion of $\eta_m = \eta_m(x_m)$ is*

$$\eta_m = \begin{cases} \left(1+O(k^{-1})\right)2ik\,(-1)^m\, e^{ik\varphi_m}\mu_m & \text{on the illuminated region of } \partial K_m \\ O(k^{-\infty}) & \text{on the shadow region of } \partial K_m \end{cases}$$

where $\mu_0 = \alpha\cdot\nu_0$, and for $m = 1,2,\dots$

$$\mu_m = \frac{x_m - x_{m-1}}{|x_m - x_{m-1}|}\cdot\nu_m \left(\prod_{i=1}^m A_i\right)^{-1/2}.$$

Here the A_i's are defined recursively as

$$A_1 = 1 + \frac{2\kappa_0|x_1-x_0|}{\frac{x_1-x_0}{|x_1-x_0|}\cdot\nu_0}, \quad \text{and} \quad A_{i+1} = 1 + \frac{2\kappa_i|x_{i+1}-x_i|}{\frac{x_{i+1}-x_i}{|x_{i+1}-x_i|}\cdot\nu_i} + \frac{|x_{i+1}-x_i|}{|x_i-x_{i-1}|}\left(1-\frac{1}{A_i}\right) \text{ for } i = 1,\dots,m-1.$$

Decay of Multiple-scattering Iterates for Trapping Obstacles

We now consider a trapping orbit $\{\partial K_m\}_{m\geq 0}$ with period p, that is $\partial K_m = \partial K_{m+p}$ for all m. Let $(a_1,\ldots,a_p) \in \partial K_1 \times \cdots \times \partial K_p$ be the *unique* p-tuple *minimizing* $\varphi(x_1,\ldots,x_p) = \left|x_p - x_1\right| + \sum_{m=1}^{p-1} |x_{m+1} - x_m|$, $(x_1,\ldots,x_p) \in \partial K_1 \times \cdots \times \partial K_p$. Our second main result [3] shows that, for $m \gg 1$,

$$\left\| \frac{\eta_{m+p}(x)}{\eta_m(x)} - (-1)^p e^{ik\varphi(a_1,\ldots,a_p)} \left(\prod_{j=1}^{p} L_j \right)^{-1/2} \right\|_{L_\infty(\partial K_m)} = O(k^{-1}) \tag{6}$$

where L_j are the solutions of quadratic equations depending only and explicitly on the geometrical properties of the obstacles $K_1,\ldots,K_p$ precisely at the points $a_1,\ldots,a_p$.

Numerical Results

Here we provide two sets of numerical experiments exemplifying our rate of decay formula (6) over periodically trapping orbits. In Figures 1 and 2, where $p = 2$ and $p = 3$ respectively, the left pane provides the corresponding geometrical configuration; the middle pane displays the left-hand side of (6) on ∂K_0 for $m = 0, p, \ldots, 25p$ where $K_j = K_{p+j} = \ldots = K_{25p+j}$ $(0 \leq j \leq p-1)$; the right pane, on the other hand, displays the same quantity against $\log_{10} k$ for $m = 25p$. As is apparent then, our rate of decay formula (6) does not depend on the direction of incidence, and the error involved in any sufficiently large and finite number of reflections is of order k^{-1}.

References

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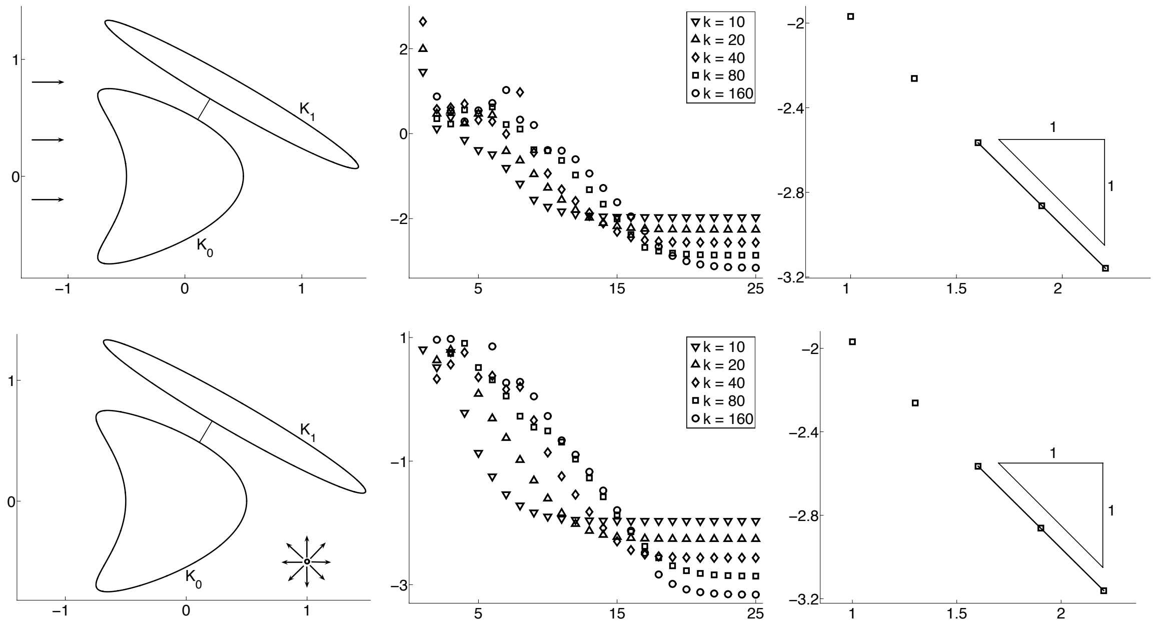

Figure 1: Two-periodic configurations *with occlusion* and *non-convex* scatterers; top: plane-wave illumination from left; bottom: point-source illumination (located at $[1,-0.5]$).

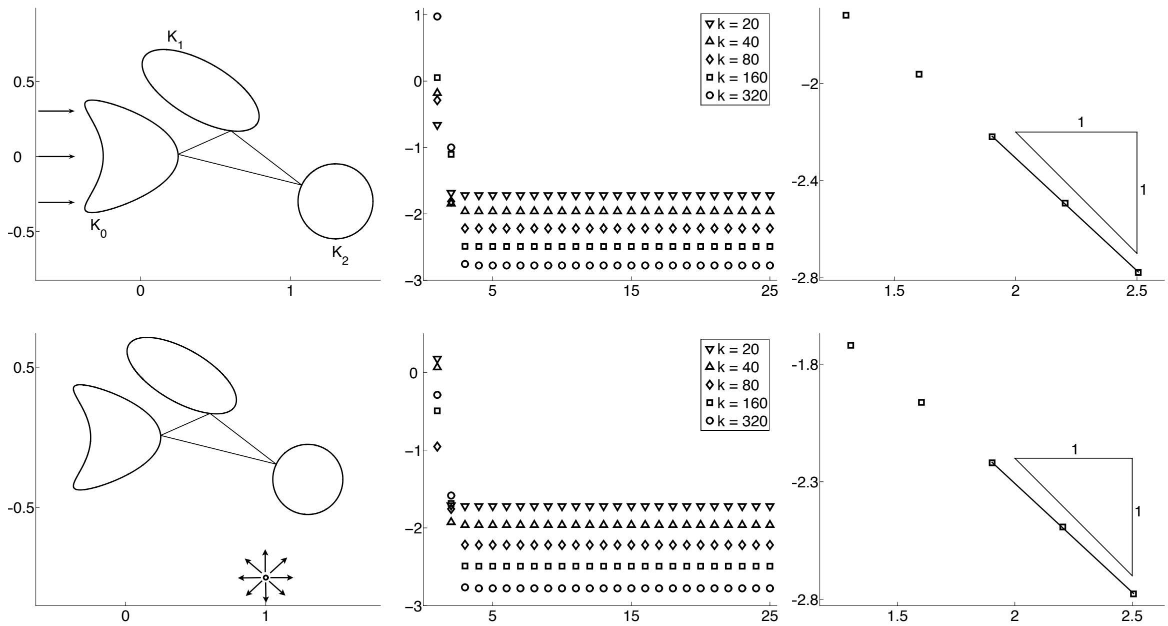

Figure 2: Three-periodic configurations *with occlusion* and *non-convex* scatterers; top: plane-wave illumination from the left; bottom: point-source illumination (located at $[1,-0.8]$).