A HIGH-FREQUENCY INTEGRAL EQUATION METHOD FOR ELECTROMAGNETIC AND ACOUSTIC SCATTERING SIMULATIONS : RATE OF CONVERGENCE OF MULTIPLE SCATTERING ITERATIONS

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Abstract

We present an analysis of a recently proposed integral equation method for the solution of high-frequency electromagnetic and acoustic scattering problems that delivers error-controllable solutions in frequency-independent computational times. Within single scatterer configurations the method is based on the use of an appropriate ansatz for the unknown surface densities and on suitable extensions of the method of stationary phase. Extension to multiple-scattering configurations, in turn, is attained through consideration of an iterative (Neumann) series that successively accounts for multiple reflections. Here we show that the convergence properties of this series in the high-frequency regime depend solely on geometrical characteristics. Moreover, for periodic orbits, we explicitly determine the convergence rate in the limit of vanishing wavelength, and we present some numerical results that confirm it as an accurate estimate for finite frequencies.

Introduction

Over the last two decades, accurate and efficient direct numerical schemes have been developed and successfully applied to the simulation of electromagnetic and acoustic wave propagation. However, all of these methods require the resolution of wavelength, and this restricts their applicability to moderately low frequencies. For higher frequencies, accordingly, the only practical recourse is to resort to asymptotic methods (e.g. ray tracing) as these by-pass the need for frequency-dependent discretizations. These methods, on the other hand, are not error-controllable since they solve an approximate model instead of the original equations (e.g. the eikonal equation instead of the Helmholtz equation or the Maxwell system).

Recently, an integral equation method featuring error-controllability and frequency-independent discretizations has been proposed for surface scattering problems in the high-frequency regime [1], [2]. Within single scattering configurations the method is based on the use of an appropriate ansatz for the unknown surface densities and on suitable extensions of the method of stationary phase. Extension to multiple-scattering configurations, in turn, is attained through consideration of an iterative (Neumann) series that successively accounts for multiple reflections. Here we show that the convergence properties of this series in the high-frequency regime depend solely on geometrical characteristics. Moreover, for periodic orbits, we explicitly determine the convergence rate in the limit of vanishing wavelength, and we present a variety of numerical results that confirm it as an accurate estimate for finite frequencies.

Integral Equations and Multiple Scattering

We consider the problem of evaluating the scattering of an incident plane wave $u^{inc}(x) = e^{ik\alpha\cdot x}$, $|\alpha| = 1$, from a bounded obstacle Ω. For the sake of brevity, we restrict ourselves to the two-dimensional context for which the relevant frequency-domain problem is modelled by the Helmholtz equation

$$\Delta u(x) + k^2 u(x) = 0, \quad x \in \mathbf{R}^2 \backslash \overline{\Omega}; \qquad (1)$$

for definiteness, we assume Dirichlet boundary conditions (TE polarization in electromagnetics)

$$u(x) = -u^{inc}(x), \quad x \in \partial\Omega. \qquad (2)$$

An integral equation formulation of (1), (2) is given by

$$\eta - R\eta = 2\partial u^{inc}/\partial\nu, \quad \text{on } \partial\Omega. \qquad (3)$$

Here ν is the outward unit normal to $\partial\Omega$, $\eta = \partial u/\partial\nu$ is the *surface current*, and

$$R\eta(x) = -2\int_{\partial\Omega} \frac{\partial\Phi(x,y)}{\partial\nu(x)}\eta(y)ds(y)$$

where Φ is the radiating free-space Green function.

When $\Omega = (\Omega_i)_{i=1,\ldots,N}$ is a finite union of disjoint sets, equation (3) takes on the coordinate form

$$\eta_i - R_{ii}\eta_i - \sum_{j\neq i} R_{ij}\eta_j = f_i \qquad (4)$$

where $\eta_i = \eta|_{\partial\Omega_i}$, $f_i = (2\partial u^{inc}/\partial\nu)|_{\partial\Omega_i}$, and on $\partial\Omega_i$

$$R_{ij}\eta_j(x) = -2\int_{\partial\Omega_j} \frac{\partial\Phi(x,y)}{\partial\nu(x)}\eta_j(y)ds(y).$$

The diagonal operators R_{ii} correspond precisely to the scattering problems for each isolated sub-surface and are therefore invertible (away from internal resonances). Then, with $\eta = [\eta_1\ \eta_2\ ...\ \eta_N]^T$, $A = [A_{ij}]$ and $g = [g_1\ g_2\ ...\ g_N]^T$, equation (4) can be written as

$$(I - A)\eta = g \quad \text{on } \partial\Omega \tag{5}$$

where $A_{ij} = (I - R_{ii})^{-1} R_{ij}$ if $i \neq j$, $A_{ii} = 0$, and $g_i = (I - R_{ii})^{-1} f_i$. The series solution to (5) is given by

$$\eta = \sum_{m=0}^{\infty} A^m g \quad \text{on } \partial\Omega.$$

At this stage, we note that $[A^0 g]_i = g_i$, and for $m \geq 1$

$$[A^m g]_i = \sum_{j_{m-1} \neq i} A_{i j_{m-1}} A_{j_{m-1} j_{m-2}} \cdots A_{j_1 j_0} g_{j_0}.$$

Here the summation is taken over all obstacle paths $\Omega_{j_0}, \Omega_{j_1}, \cdots, \Omega_{j_{m-1}}$ where no two consecutive objects are the same. Evidently then, the total surface current η is the superposition over all finite obstacle paths of the iterated currents arising from multiple reflections. Thus, given an obstacle path, that is a sequence $(\Omega_m)_{m \geq 0}$ where no two consecutive objects are the same, one needs to solve the integral equations

$$\eta_0 - R_{0,0}\eta_0 = f_0, \quad \text{on } \partial\Omega_0 \tag{6}$$

and inductively for $m = 1, 2, \cdots$

$$\eta_m - R_{m,m}\eta_m = R_{m,m-1}\eta_{m-1}, \quad \text{on } \partial\Omega_m. \tag{7}$$

The significance of this interpretation stems from the fact that it guarantees that each of the problems in (6), (7) entails the solution of problems within single scattering configurations for which the methods described in [1], [2] provide an error-controllable scheme with fixed computational complexity.

Convergence of the Iterated Series

Suppose that the obstacles $(\Omega_i)_{i=1,\cdots,N}$ are convex, and are *visible* in the sense that no Ω_i meets with the convex hull of any other pair of obstacles. For a fixed m, and a fixed $x_m \in \partial\Omega_m$, $(x_0, \cdots, x_{m-1}) \in \partial\Omega_0 \times \cdots \times \partial\Omega_{m-1}$ will denote the unique set of points determined by the geometrical optics solution. We also set $\nu_m := \nu(x_m)$, and $\varphi_m = \varphi_m(x_m) = \alpha \cdot x_0 + \sum_{i=0}^{m-1} |x_{i+1} - x_i|$. Our first result states that, in the high-frequency regime, the behavior of the currents, that is the solutions $\eta_0, \eta_1, \cdots, \eta_m, \cdots$ of (6), (7), depends solely on the geometrical characteristics of the surfaces $\partial\Omega_i$ on the optical ray paths.

Theorem (*Asymptotic Representations of the Iterated Currents*) At each reflection $m = 0, 1, \cdots$, the asymptotic representation of the iterated currents $\eta_m = \eta_m(x_m)$ are given, on proper compact subsets of the illuminated regions, by

$$\eta_m \left(1 + \mathcal{O}\left(k^{-1}\right)\right) = 2ik\,(-1)^m\, e^{ik\varphi_m} \mu_m$$

where $\mu_0 = \alpha \cdot \nu_0$, and for $m = 1, 2, \cdots$

$$\mu_m = \frac{x_m - x_{m-1}}{|x_m - x_{m-1}|} \cdot \nu_m \left(\prod_{i=1}^{m} A_i\right)^{-1/2}.$$

Here the A_i's are defined recursively as

$$A_1 = 1 + \frac{2\kappa_0\,|x_1 - x_0|}{\frac{x_1 - x_0}{|x_1 - x_0|} \cdot \nu_0}$$

and for $i = 1, \cdots, m-1$

$$A_{i+1} = 1 + \frac{2\kappa_i\,|x_{i+1} - x_i|}{\frac{x_{i+1} - x_i}{|x_{i+1} - x_i|} \cdot \nu_i} + \frac{|x_{i+1} - x_i|}{|x_i - x_{i-1}|}\left(1 - \frac{1}{A_i}\right).$$

Moreover, at each reflection m, the iterated current η_m vanishes to first order on proper compact subsets of the shadowed region.

The proof of this theorem is based on the classical methods for the evaluation of oscillatory integrals.

We now consider an orbit $(\partial\Omega_m)_{m \geq 0}$ with period *p*, that is $\partial\Omega_m = \partial\Omega_{m+p}$ for all *m*. Let $(a_1, \cdots, a_p) \in \partial\Omega_1 \times \cdot \times \partial\Omega_p$ be the *unique* p-tuple *minimizing* the phase $\varphi(x_1, \cdots, x_p) = |x_p - x_1| + \sum_{m=1}^{p-1} |x_{m+1} - x_m|$, $(x_1, \cdots, x_p) \in \partial\Omega_1 \times \cdots \times \partial\Omega_p$. Our next result (whose proof is based on a detailed analysis of ray paths and the use of the theory of "limit p-periodic continued fractions" [3]) shows that the periodic ratio of iterated currents converges exponentially and uniformly to a quantity determined only by the points $(a_1, \cdots, a_p)$.

Theorem (*Rate of Convergence over Periodic Orbits*) As the number of reflections m tends to infinity,

$$\frac{\mu_{m+p}(x)}{\mu_m(x)} = r_\infty + \mathcal{O}\left(\frac{\sqrt{m}}{r^{m/p}}\right)$$

uniformly for x in any proper compact subset of $\bigcup_{n=1}^{p} \partial\Omega_n^I$ where $\partial\Omega_n^I$ are the limiting illuminated regions. Here r_∞ depends only and explicitly on the distances $|a_{m+1} - a_m|$, the curvatures κ_m at the points a_m

of the surfaces $\partial\Omega_m$ and the scalar products $\frac{a_{m+1}-a_m}{|a_{m+1}-a_m|} \cdot \nu(a_m)$. In particular, for $p = 2$,

$$r_\infty = \left(\gamma \left[1 + \sqrt{1 - \frac{1}{\gamma}} \; \right] \right)^{-1/2} \qquad (8)$$

where $d = |a_2 - a_1|$ and $\gamma = (1 + \kappa_1 d)(1 + \kappa_2 d)$.

Numerical Examples

Here we provide two numerical experiments exemplifying our theorem on the rate of convergence over periodic orbits, and we show that this rate can provide an accurate estimate for finite frequencies.

First we consider a two-periodic configuration consisting of two ellipses illuminated from the left (see Figure 1). Figure 2 shows that the rate of convergence in (8) is, indeed, a very good approximation even at very low frequencies.

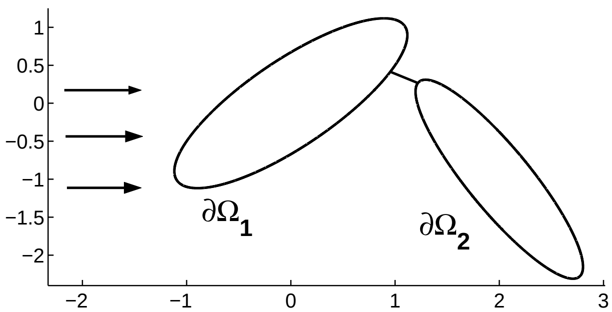

Figure 1: A two-periodic configuration

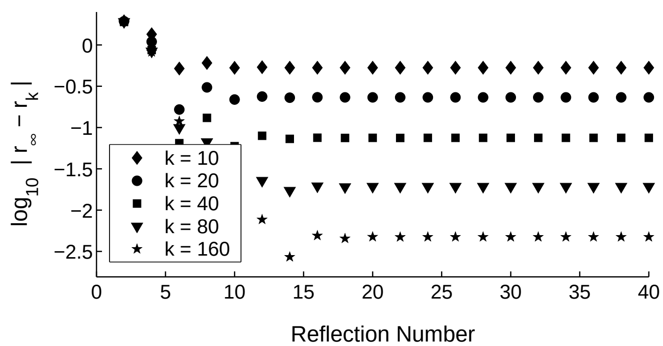

Figure 2: Logarithmic plot of the error $|r_\infty - r_k|$ where $r_k = \eta_{m+2}/\eta_m$ for the corresponding wavenumber k

Finally we consider a three-periodic configuration consisting of three ellipses illuminated from the top (see Figure 3). As we mentioned, an explicit formula analogous to (8) (though significantly complicated) can be derived in this case. A comparison of this rate with that attained at finite frequencies in this case is displayed in Figure 4.

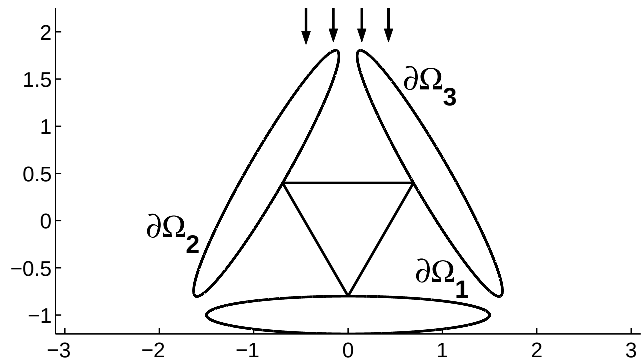

Figure 3: A three-periodic configuration

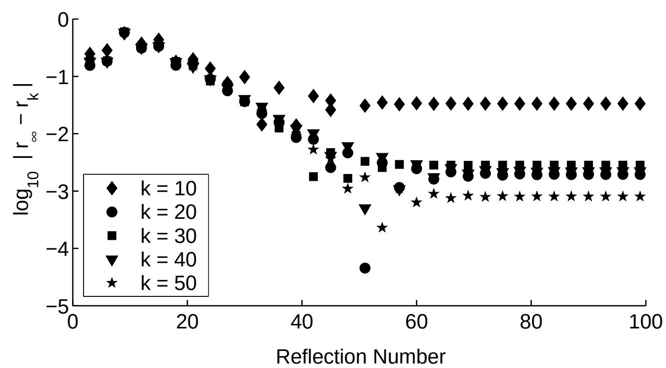

Figure 4: Logarithmic plot of the error $|r_\infty - r_k|$ where $r_k = \eta_{m+3}/\eta_m$ for the corresponding wavenumber k

References

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