Convergent scattering algorithms

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Hybrid numerical methods based upon a combination of integral equations and asymptotic theories for the solution of high-frequency scattering problems have found an increased interest within the last two decades. Indeed, the methodologies developed in this time span, that specifically concern scattering off a single two-dimensional smooth convex obstacle [2, 5, 6, 8], display the capability of predicting scattering returns within any prescribed accuracy utilizing a number of degrees of freedom independent of (or only mildly dependent on) the frequency.

This report concerns (i) the classification of Hörmander classes and asymptotic expansions of multiple scattering iterations for a collection of smooth convex obstacles that thereby allow for the extension of the single-scattering solvers in [5, 6, 8] to multiple-scattering configurations to accompany the algorithm in [3]; and (ii) the derivative estimates of multiple scattering iterations that are necessary for their rigorous numerical analysis and that facilitate the development of convergent scattering algorithms (for each fixed value of the wavenumber k) for the computation of each iterate (utilizing a number of degrees of freedom that depends only mildly on the frequency to attain a prescribed accuracy) based on the ideas in [5].

To present a summary of the relevant results we have recently developed in [1, 7], let us consider the problem of evaluating the scattering of an incident plane wave $u^{\mathrm{inc}}(x) = e^{ik\alpha\cdot x}$, $|\alpha| = 1$, from a compact impenetrable obstacle K with a smooth boundary ∂K. Throughout this note we concentrate on two-dimensional configurations wherein the relevant frequency-domain problem is modeled by the Helmholtz equation

$$\Delta u(x) + k^2 u(x) = 0, \quad x \in \mathbb{R}^2 \backslash K,$$

where the scattered field u is required to satisfy the Sommerfeld radiation condition [4]; here, for definiteness, we shall assume Dirichlet boundary conditions on ∂K.

As is well known, this problem can be restated in the form of an integral equation in a variety of ways [4]; a convenient form for our purposes is that derived from the Green identities resulting in the equation

$$\eta(x) - \int_{\partial K} \frac{\partial G(x,y)}{\partial \nu(x)}\, \eta(y)\, ds(y) = 2\, \frac{\partial u^{\mathrm{inc}}(x)}{\partial \nu(x)}, \quad x \in \partial K \tag{1}$$

for the unknown density η (the normal derivative of the total field), where $\nu(y)$ denotes the vector normal to ∂K and exterior to K,

$$\Phi(x,y) = \frac{i}{4}\, H_0^{(1)}(k|x-y|)$$

is the *outgoing* Green function, and $G = -2\Phi$. Since the solution of the integral equation (1) is not unique when the wavenumber k is an internal resonance, in practical implementations a "combined field" integral equation formulation must be used [4]. For the sake of simplicity, the derivations that follow, for the description of multiple scattering formulation of the scattering problem, are based upon the integral equation (1).

Let us now further suppose that the sound-soft obstacle K is decomposed into a finite collection of disjoint compact sub-scatterers $K = \bigcup_{\sigma\in\mathcal{I}} K_\sigma$. Then the integral equation (1) can be written as

$$(I - R)\,\eta = f \tag{2}$$

where $\eta(x) = (\eta_{\sigma_1}(x), \ldots, \eta_{\sigma_{|\mathcal{I}|}}(x))^t$ and $f(x) = (f_{\sigma_1}(x), \ldots, f_{\sigma_{|\mathcal{I}|}}(x))^t$ with η_σ and f_σ defined on ∂K_σ and

$$f_\sigma(x) = 2ik\, e^{ik\alpha\cdot x}\, \alpha\cdot\nu(x),$$

and the operator R is defined as

$$(R_{\sigma\tau}\eta_\tau)(x) = \int_{\partial K_\tau} \frac{\partial G(x,y)}{\partial\nu(x)}\, \eta_\tau(y)\, ds(y), \qquad x \in \partial K_\sigma.$$

Inverting the diagonal part of (2) yields the equivalent relation

$$(I - T)\eta = g \tag{3}$$

with

$$g_\sigma = (I - R_{\sigma\sigma})^{-1} f_\sigma, \qquad \sigma\in\mathcal{I}$$

and

$$T_{\sigma\tau} = \begin{cases} (I - R_{\sigma\sigma})^{-1} R_{\sigma\tau} & \text{if } \sigma\neq\tau \\ 0 & \text{otherwise.} \end{cases}$$

The formulation (3) provides a convenient mechanism to account for multiple scattering since the m-th term in its Neumann series solution

$$\eta = \sum_{m=0}^{\infty} \eta^m = \sum_{m=0}^{\infty} T^m g \tag{4}$$

corresponds to contributions arising as a result of waves that have undergone m reflections. More precisely, we have

$$\eta^m\big|_{\partial K_\sigma} = \sum_{\substack{\tau_0,\cdots\tau_{m-1}\in\mathcal{I} \\ \sigma\neq\tau_{m-1},\tau_j\neq\tau_{j-1}}} T_{\sigma\tau_{m-1}} T_{\tau_{m-1}\tau_{m-2}} \cdots T_{\tau_1\tau_0}\, g_{\tau_0}, \tag{5}$$

where each application of a $T_{\sigma\tau}$ entails an evaluation on ∂K_σ of a field generated by a current on ∂K_τ, and its use as an incidence for a subsequent solution of a single-scattering problem on ∂K_σ. Accordingly, equations (4) and (5) guarantee that η can be recovered as the superposition (over all infinite paths $\{K_m\}_{m\geq 0} \subset \{K_\sigma : \sigma\in\mathcal{I}\}$) of multiple scattering iterations η_m that recursively solve the integral equations

$$\eta_0(x) - \int_{\partial K_0} \frac{\partial G(x,y)}{\partial\nu(x)}\, \eta_0(y)\, ds(y) = 2\,\frac{\partial u^{\mathrm{inc}}(x)}{\partial\nu(x)}\,, \quad x\in\partial K_0$$

and, for $m \geq 1$,

$$\eta_m(x) - \int_{\partial K_m} \frac{\partial G(x,y)}{\partial \nu(x)} \eta_m(y)\, ds(y) = \int_{\partial K_{m-1}} \frac{\partial G(x,y)}{\partial \nu(x)} \eta_{m-1}(y)\, ds(y), \quad x \in \partial K_m$$

on the path $\{K_m\}_{m \geq 0}$.

Supposing now that the obstacles K_σ are strictly convex, under certain conditions, the multiple-scattering iterations η_m admit the factorizations

$$\eta_m(x) = e^{ik\varphi_m(x)}\, \eta_m^{\text{slow}}(x), \quad x \in \partial K_m \tag{6}$$

wherein φ_m is the m-th geometrical optics phase, and where the asymptotic properties of the slow envelope η_m^{slow} are as follows (see [1, 7] for details):

Theorem 1 (Hörmander classes and asymptotic expansions of η_m^{slow}, [1, 7]) *The asymptotic characteristics of the slow densities η_m^{slow} specified by (6) are as follows:*
(i) *On the m-th illuminated region ∂K_m^{IL}, $\eta_m^{\text{slow}}(x) = \eta_m^{\text{slow}}(x,k)$ belongs to the Hörmander class $S^1_{1,0}(\partial K_m^{IL} \times (0,\infty))$ and admits the asymptotic expansion*

$$\eta_m^{\text{slow}}(x,k) \sim \sum_{j \geq 0} k^{1-j} a_{m,j}(x)$$

where $a_{m,j}(x)$ are complex-valued C^∞ functions. Accordingly, for any $N \in \mathbb{N} \cup \{0\}$, the difference

$$r_{m,N}(x,k) = \eta_m^{\text{slow}}(x,k) - \sum_{j=0}^{N} k^{1-j} a_{m,j}(x)$$

belongs to $S^{-N}_{1,0}(\partial K_m^{IL} \times (0,\infty))$ and thus satisfies the estimates

$$\left| D_x^\beta D_k^n r_{m,N}(x,k) \right| \leq C_{m,\beta,n,S}(1+k)^{-N-n}$$

on any compact subset S of ∂K_m^{IL} for any multi-index β and $n \in \mathbb{N} \cup \{0\}$.
(ii) *Over the entire boundary ∂K_m, $\eta^{\text{slow}}(x,k)$ belongs to $S^1_{2/3,1/3}(\partial K_m \times (0,\infty))$ and admits the asymptotic expansion*

$$\eta_m^{\text{slow}}(x,k) \sim \sum_{p,q \geq 0} k^{2/3-2p/3-q}\, b_{m,p,q}(x) \Psi^{(p)}(k^{1/3} Z_m(x))$$

where $b_{m,p,q}(x)$ are complex-valued C^∞ functions, $Z_m(x)$ is a real-valued C^∞ function that is positive on the illuminated region ∂K_m^{IL}, negative on the shadow region ∂K_m^{SR}, and vanishes precisely to first order on the shadow boundary ∂K_m^{SB}, and the function Ψ is a certain contour integral of an Airy function (see [9]*). Note specifically then, for any $P, Q \in \mathbb{N} \cup \{0\}$, the difference*

$$R_{m,P,Q}(x,k) = \eta_m^{\text{slow}}(x,k) - \sum_{p,q=0}^{P,Q} k^{2/3-2p/3-q}\, b_{m,p,q}(x) \Psi^{(p)}(k^{1/3} Z_m(x))$$

belongs to $S^{-\mu}_{2/3,1/3}(\partial K_m \times (0,\infty))$, $\mu = \min\{2P/3, Q\}$, and thus satisfies the estimates

$$\left| D_x^\beta D_k^n R_{m,P,Q}(x,k) \right| \leq C_{m,\beta,n}(1+k)^{-\mu-2n/3+|\beta|/3}$$

for any multi-index β and $n \in \mathbb{N} \cup \{0\}$.

As we anticipated, the preceding theorem provides the necessary theoretical background for the extension of the single-scattering solvers [6, 6, 8] to multiple scattering configurations to accompany the algorithm in [3]. As a byproduct, we now present the derivative estimates of the slow envelopes η_m^{slow} that can be directly utilized for the numerical analysis of multiple scattering iterations η_m as is done in [5, 6] for a single convex obstacle.

Theorem 2 (Derivative estimates of η_m^{slow}, [7]) *Let $m \geq 0$, and denote by $y(s) = (y^1(s), y^2(s))$ the arc-length parametrization of ∂K_m. Then, for all $n \in \mathbb{N} \cup \{0\}$, there exist a constant $C_n > 0$ independent of k and s such that for all k sufficiently large,*

$$\left|D_s^n \eta_m^{\text{slow}}(y(s))\right| \leq k \begin{cases} C_n, & n = 0, 1, \\ C_n \left[1 + \sum_{j=2}^n k^{(j-1)/3}(1 + k^{1/3}|w(s)|)^{-(j+2)}\right], & n \geq 2, \end{cases}$$

where $w(s) = (s-a)(b-s)$ and $\partial K_m^{SB} = \{y(a), y(b)\}$ is the set of m-th shadow boundary points.

References

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