High-frequency multiple scattering problems: An appropriate preconditioner for a Krylov subspace algorithm

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Abstract

This paper is devoted to iterative methods dealing with high frequency multiple scattering problems. The analysis of the rate of convergence performed in [4], [1] suggests that the effective convergence depends on the geometrical configuation of the obstacles. We investigate in this paper an appropriate Krylov subspace method combined with suitable preconditioner based on the Kirchhoff approximations to significantly decrease the number of iterations.

Introduction

To extend the high-frequency integral equation introduced in [2] to multiple scattering configurations, a geometrical optics solver (as post-process) is applied to evaluate the phase at each reflection. The solution is then reduced to an iterative procedure defined by the Neumann series. The analysis of the corresponding rate of convergence shows that the effectiveness of this technique depends on the configuration of the obstacles [4], [1]. Roughly speaking, small distance between the obstacles deteriorates the convergence. To overcome this difficulty, we have introduced an appropriate Krylov subspace method that significantly decreases the number of iterations [3], [5]. Indeed, as was demonstrated in [3], [5], the knowledge of the phase is restricted only to the iterates given in the Neumann series. This prevents the utilization of standard Krylov subspace method for solutions based on the high-frequency integral equation used in this work. We have therefore modified the Krylov subspace method using an identification process that allows us to avoid this problem. To improve this overall algorithm, we investigate in this paper a suitable preconditioner based on the Kirchhoff approximations to attain an additional decrease in the number of iterations. Indeed, the geometrical optics solver mentioned above reduces the cost of this preconditioner only to application of the stationary phase method to non-singular integrals.

Integral formulation

Consider a sound soft acoustic scattering problem for a bounded obstacle $K \subset R^n$ with $n = 2, 3$

$$\begin{cases} \Delta + k^2 u = 0 \quad x \in R^n \backslash \overline{K}, \\ u(x) = -u^{\text{inc}} = -e^{ik\alpha\dot{x}} \quad x \in \partial K, \\ \lim_{|x|\to\infty} \left[\left(\frac{x}{|x|}, \nabla u(x) \right) - iku(x) \right] = 0. \end{cases} \tag{1}$$

The solution of this problem can be expressed under the following integral equation

$$\eta(x) - \int_{\partial K} \frac{\partial G(x,y)}{\partial \nu(x)} \eta(y) ds(y) = 2 \frac{\partial u^{\text{inc}}(x)}{\partial \nu(x)} \tag{2}$$

where η is the unknown describing the normal velocity of the total field and $G = -2\Phi$ with Φ indicating the Green function. Assume that scatterer K is decomposed into two disjoint sets $K = K_1 \cup K_2$, the formulation (2) becomes

$$(I - \mathcal{A}) \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} (I - T_{11})^{-1} f_1 \\ (I - T_{22})^{-1} f_2 \end{bmatrix}, \tag{3}$$

where

$$\mathcal{A} = \begin{bmatrix} 0 & (I - T_{11})^{-1} R_{12} \\ (I - T_{22})^{-1} R_{21} & 0 \end{bmatrix} \tag{4}$$

and

$$T_{jj}(\eta_j)(x) = 2 \int_{\partial K_j} G(x,y) \eta_j(y)\, ds(y) \quad x \in \partial K_j, \tag{5}$$

$$R_{ji}(\eta_j)(x) = 2 \int_{\partial K_i} G(x,y) \eta_i(y) ds(y) \quad x \in \partial K_j. \tag{6}$$

Here, η_j (resp. f_j), $j = 1, 2$, indicates the restriction of η (resp. $f(x)$ which is the right hand side in the equation (2)) to ∂K_j. The iterative procedure is finally defined through the following series

$$\eta = \sum_{m=0}^{\infty} \eta^m = \sum_{m=0}^{\infty} \mathcal{A}^m g. \tag{7}$$

As post-process (geometrical optics solver), the phase is evaluated as

$$\varphi_j^m(x) = \begin{cases} \varphi_j^0(x) = \alpha \cdot x & m = 0,\, x \in \partial K_j, \\ \varphi_j^0(x) + \delta_j^m(x) & m \geq 1,\, x \in \partial K_j, \end{cases} \tag{8}$$

where $\delta_j^m(x)$ is a sequence of functions measuring the optical distance travelled by a ray arriving at $x \in \partial K_j$ after m reflections and $j = 1, 2$. Knowledge of the phases, in turn, allows for their extraction and therefore the use of the integral equation described in [2]. The details and the advantages of this algorithm are described in [4], [5].

In the case presented here (two cylindrical convex structures K_1 and K_2), the rate of convergence of the Neumann series is given by

$$\mathcal{R}_k = e^{2ikd} \left(\sqrt{r} + \sqrt{r-1}\right)^{-1} \tag{9}$$

where $d = \text{dist}(K_1, K_2)$, $r = (1+d\kappa_1)(1+d\kappa_2)$ and κ_j are the curvatures at the distance minimizing points [4]. This suggests that the convergence by Neumann series is impaired when $d \to 0$. To improve it, we have introduced a new Krylov-subpace method adapted to the high-frequency aspect of the problem [3]. Applying a preconditoner leads to solving the equation

$$(I - \mathcal{K})^{-1}(I - \mathcal{A})\eta = (I - \mathcal{K})^{-1} g \tag{10}$$

and $\mathcal{K}$ represents the Kirchhoff approximations, formally given by

$$\mathcal{K} = \begin{bmatrix} 0 & \tilde{R}_{12} \\ \tilde{R}_{21} & 0 \end{bmatrix}. \tag{11}$$

where $\tilde{R}_{ij}$ are the operators (6) evaluated on the illuminated region using the stationary phase method at each iteration. To compute $(I - \mathcal{K})^{-1}$, we have used the Neumann series. The numerical example presented here consists of two unit circles centered respectively at $(0, 0)$ and $(0, -2.3)$. We have considered the wavenumber $k = 200$ and we have illuminated the obstacles by a plane wave directed along the vector $(1, 0)$. The figure depicted in 1 shows that only three iterations are needed to obtain a 10^{-4} accuracy. The fact that this error do not attain the machine precision is due to the truncation of the series used to compute the preconditioner.

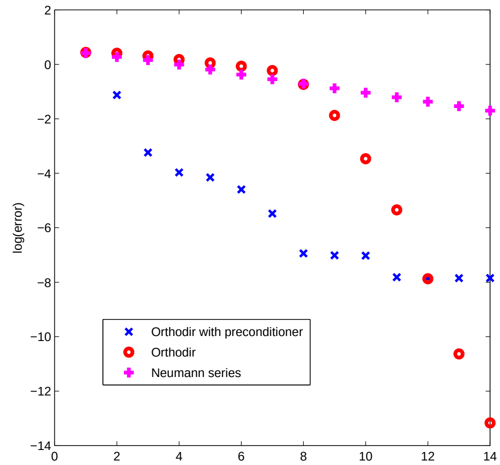

Figure 1: Number of iterations versus logarithmic plot of L^∞ error.

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