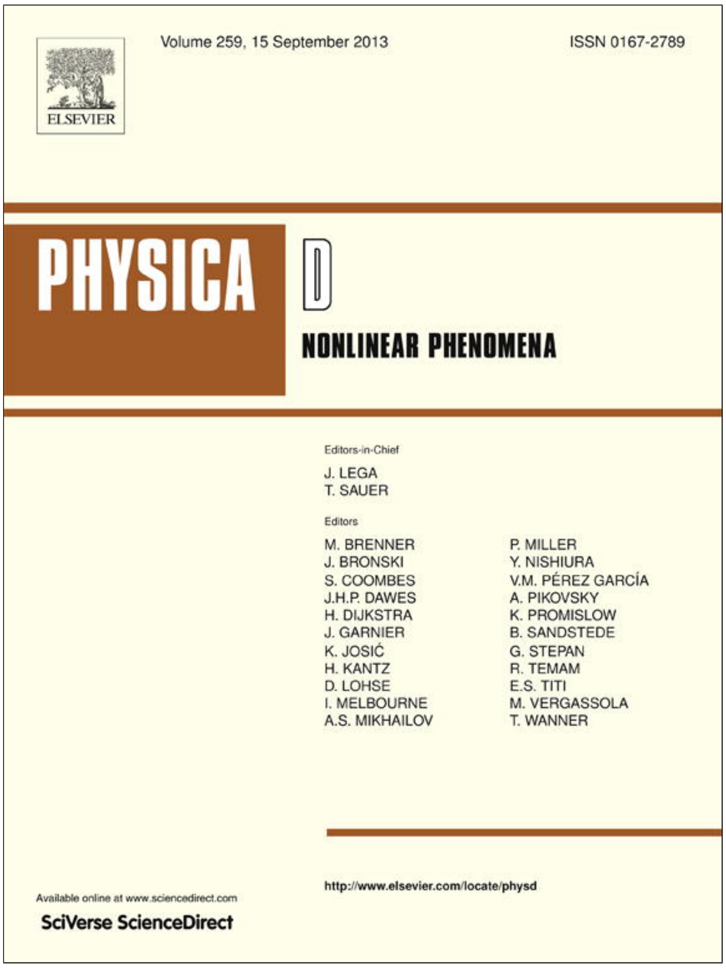
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# On the integrability of a generalized Davey–Stewartson system

A. Eden [1], T.B. Gürel [*]

*Department of Mathematics, Boğaziçi University, Bebek 34342 İstanbul, Turkey*

## HIGHLIGHTS

- The method of Zakharov and Shulman is applicable to a three-component system.
- For a class of generalized Davey–Stewartson system necessary conditions for integrability are obtained.
- This class is shown to be not integrable unless it reduces to the known integrable cases.

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ABSTRACT

In this work we investigate the integrable cases of the elliptic–hyperbolic–hyperbolic generalized Davey–Stewartson system introduced in Babaoğlu and Erbay (2004) [6] following the method of Zakharov and Shulman (1980) [3]. This method provides us with a set of algebraic conditions on the parameters of the system, which are just necessary conditions for the system to be integrable by means of the inverse scattering transform. Taking into account the constraints arising from the physical derivation of the generalized Davey–Stewartson system as described in Babaoğlu and Erbay (2004) [6], we show that this system is integrable only when it can be transformed to an integrable case of the Davey–Stewartson system.



## 1. Introduction

The Davey–Stewartson (DS) system plays a special role among the integrable systems in $2+1$ dimensions because the $2+1$ dimensional Schrödinger equation with cubic nonlinearity fails to be integrable. In fact, when the DS system is considered with various parameters in it, there are only two integrable cases which are known as DS I and DS II; see for instance [1]. These two integrable cases are elliptic–hyperbolic and hyperbolic–elliptic type. Shulman in [2] treated a more general class of equations:

$$iu_t + L_1 u + \psi u = 0, \qquad L_2\psi = L_3|u|^2,$$

where $u$ is complex-valued, $\psi$ is real-valued and

$$L_n = \sum_{j,k=1}^{N} c_{jk}^n \frac{\partial^2}{\partial x_j \partial x_k}, \quad n = 1, 2, 3$$

are second order partial differential operators with real constant coefficients $c_{jk}^n$, and $N$ is the spatial dimension.

Shulman considered the $N = 2$ case in detail which encompasses the DS type systems among others and derived necessary conditions for integrability via inverse scattering transform. He used the method introduced in [3] and showed that there are four integrable cases in this class which are DS I and DS II systems and the systems (4) and (28) of his paper [2], provided that $L_1$ is non-degenerate. This list was given up to linear transformations of the coordinates $x_1$ and $x_2$. We should also note that these integrable systems arising from this general class are distinct by construction. Namely, the integrable system (4) in [2] can be obtained when $L_2$ and $L_3$ are both hyperbolic and have a common factor so that $L_2\psi = L_3|u|^2$ reduces to a first order equation. The system (28) in [2] corresponds to hyperbolic–hyperbolic $L_2$ and $L_3$ operators. The integrable DS I and DS II systems, on the other hand, are consequences of elliptic–hyperbolic and hyperbolic–elliptic choices of $L_2$ and $L_3$ operators, respectively. In all these cases the operator $L_1$ has the character of $L_3$. The integrable cases of the DS system were known prior to Shulman's work and indeed this fact was one of his main motivations for investigating such a general family using the method developed in [3].

* Corresponding author. Tel.: +90 212 3596651; fax: +90 212 2877173.
*E-mail addresses:* eden@boun.edu.tr (A. Eden), bgurel@boun.edu.tr, t.burak.gurel@gmail.com (T.B. Gürel).

[1] Currently on leave at the Department of Mathematics, Indiana University, Bloomington, IN 47405, USA.

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In the same paper [2], Shulman investigated the above system for $N = 3$ and concluded that there are no integrable systems of this kind if $L_1$ is nondegenerate. In the case when $L_1$ is degenerate, he concluded that only those systems which are reducible to integrable systems of the $N = 2$ case are integrable.

The two-component DS system was first derived by Davey and Stewartson in [4] and later by Djordjević and Redekopp in [5] as a model for the weakly nonlinear packets of waves propagating on the surface of water in the $x$-direction whilst the variations in the $y$-direction are more gradual. In order to incorporate the effect of the second spatial coordinate Babaoğlu and Erbay derived in [6] a three-component DS system for amplitudes of a short transverse wave, a long transverse wave and a long longitudinal wave that travel in quadratically nonlinear elastic medium:

$$\begin{aligned} &iU_\tau + pU_{\xi\xi} + rU_{\eta\eta} = q|U|^2U + \frac{k}{2\omega}(\gamma_3\phi_{1,\xi} + \gamma_1\phi_{2,\eta})U \\ &(c_g^2 - c_1^2)\phi_{1,\xi\xi} - c_2^2\phi_{1,\eta\eta} - (c_1^2 - c_2^2)\phi_{2,\xi\eta} = \gamma_3k^2(|U|^2)_\xi \\ &(c_g^2 - c_2^2)\phi_{2,\xi\xi} - c_1^2\phi_{2,\eta\eta} - (c_1^2 - c_2^2)\phi_{1,\xi\eta} = \gamma_1k^2(|U|^2)_\eta, \end{aligned} \tag{1}$$

where $\xi$ and $\eta$ are spatial coordinates, $\tau$ is the time, $U$ is the complex amplitude of the short transverse wave mode, and $\phi_1$ and $\phi_2$ are the long longitudinal and long transverse wave modes, respectively. Detailed description of the coefficients appearing in (1) is given in [6]. We mention here only some information about them that will be needed later in this study:

$$c_1^2 - 2c_2^2 > 0, \qquad \gamma_3 > \gamma_1 > 0, \qquad c_g > c_1 > c_2.$$

In addition to the above inequalities we need to state the following:

$$\begin{aligned} p &= -\frac{1}{2\omega}(c_g^2 - c_2^2 - 24m^2c_2^2k^2) \\ r &= \frac{c_2^2}{2\omega}(1 + 8m^2k^2) \\ q &= \frac{k^6\gamma_3^2}{\omega D_1(2k, 2\omega)} \end{aligned} \tag{2}$$

where $\omega = c_2k(1 + 4m^2k^2)^{1/2}$ and $D_1(k, \omega) = \omega^2 - c_1^2k^2 - 4(1 + \nu)c^2m^2k^4$ in which $m, \nu$ are positive, and $k$ is the wave number and $\omega$ is the frequency. In (1), $c_g$ is the group speed of transverse waves, $c_1$ and $c_2$ are phase speeds of longitudinal and transverse waves, respectively, and the coefficients $p$, $q$ and $r$ are all positive.

Babaoğlu and Erbay called these coupled equations generalized Davey–Stewartson equations since under some parameter regimes they can be transformed to the DS system via a nonlinear transformation. This point will be clarified in the next section.

Classification of the generalized Davey–Stewartson (GDS) system is more involved and consists of eight possible cases. Regarding the coefficients purely mathematically, this classification is given in [7]. Most of the mathematical work is done in the elliptic–elliptic–elliptic case; see [8] and the references therein. However, the elliptic–hyperbolic–hyperbolic case is physically the most relevant as pointed out in [6].

Let us remark that even for the elliptic–hyperbolic case of the DS system the mathematical analysis of the problem is quite difficult. The only result known to us about the well-posedness of the problem without smallness conditions on the initial datum is due to Hayashi [9], in which he utilized weighted Sobolev spaces. However, the literature which assumes small initial data, even for global existence, is much richer; see [10–13].

For later purposes, we briefly recall how the parameters in the GDS system (1) can be nondimensionalized and the system can be rewritten in a nonphysical form; see [14]. Defining the new independent variables $t, x, y$ that are obtained from the scalings of the old ones via

$$\tau = k^4t, \qquad \xi = \frac{k^2a_1\sqrt{r}}{c_2}x, \qquad \eta = k^2\sqrt{r}y,$$

and the new dependent variables $u, \varphi_1, \varphi_2$ via

$$U = \frac{c_2a_2}{\gamma_1a_1k^2}u, \qquad \phi_1 = \frac{a_2\sqrt{r}}{a_1\gamma_1}\varphi_1, \qquad \phi_2 = \frac{\sqrt{r}}{\gamma_1}\varphi_2,$$

where $a_1 = (c_g^2 - c_1^2)^{1/2}$ and $a_2 = (c_g^2 - c_2^2)^{1/2}$, the physical form (1) becomes

$$\begin{aligned} &iu_t + \gamma u_{xx} + u_{yy} = \chi|u|^2u + b(\alpha\varphi_{1,x} + \varphi_{2,y})u \\ &\varphi_{1,xx} - \varphi_{1,yy} - \beta\varphi_{2,xy} = \alpha(|u|^2)_x \\ &\varphi_{2,xx} - \lambda\varphi_{2,yy} - \beta\varphi_{1,xy} = (|u|^2)_y, \end{aligned} \tag{3}$$

where $\gamma$ and $\lambda$ are positive numbers, and $u$ is complex-valued.

More precisely, the set of new coefficients in the system (3) are related to the old coefficients seen in (1) by means of the following:

$$\begin{aligned} &\gamma = \frac{pc_2^2}{ra_1^2}, \qquad \chi = \frac{qc_2^2a_2^2}{\gamma_1^2a_1^2}, \qquad b = \frac{k^4}{2\omega}, \\ &\alpha = \frac{\gamma_3c_2a_2}{\gamma_1a_1^2}, \qquad \beta = \frac{c_1^2 - c_2^2}{c_2a_2}, \qquad \lambda = \frac{c_1^2a_1^2}{c_2^2a_2^2}. \end{aligned} \tag{4}$$

It is crucial to notice the difference between (1) and (3). The latter represents the elliptic–hyperbolic–hyperbolic GDS system in a mathematical manner in the sense that the relations among parameters are not visible, whilst the former bears the constraints occurring as a result of the underlying physical model.

In [15] a global existence, without uniqueness, the result was established for weak solutions of the elliptic–hyperbolic–hyperbolic GDS system (3) when the initial datum has small enough mass. In the present study we consider this system from integrability point of view and seek a combination of the parameters of the system for which it does not reduce to the elliptic–hyperbolic DS system and is integrable by means of inverse scattering transform. The answer turns out to be negative. Namely, we prove in the next sections the main result of this paper, that is as follows.

**Theorem 1** (*Main Result*)**.** *The GDS system* (1) *is integrable via the inverse scattering transform only when it can be reduced to an integrable DS system. Hence, no choice of parameters leads to a new integrable system which is a particular member of the system* (1)*.*

To show this statement we proceed as follows. In Section 2, we will describe a scheme that reduces the GDS system (3) to a DS system, which has already been discussed in [6]. We will only adapt this discussion to various forms of the GDS system appearing in our study. This reduction scheme turns out to be useful to decide whether a possibly integrable case of the GDS system is reducible to a DS system. In Section 3, we introduce the method of Zakharov and Shulman and apply it to the elliptic–hyperbolic–hyperbolic GDS system (3). Consequently, we obtain the necessary conditions for integrability at the lowest perturbation order. These conditions are also referred to as the conditions for the vanishing of the first vertex. We examine these necessary conditions case by case in Section 4 and compare them with the reducibility condition whenever necessary. Thus we complete the analysis of (1).

## 2. A scheme to reduce the GDS system to a DS system

Reducibility of the GDS system to a DS system is addressed in various papers such as [6,7,14]. This reduction cannot be achieved for arbitrary positions of coefficients in the GDS system. In [14] it was shown how (1) reduces to an elliptic–hyperbolic DS system. Since our treatment relies upon the elliptic–hyperbolic–hyperbolic assumption, we focus on this special case.

This issue is important for our purposes particularly because of the integrability problem. The DS system,

$$\begin{aligned} &iq_t + \tilde{\gamma} q_{xx} + q_{yy} = \tilde{\chi}|q|^2 q + \tilde{b}\psi_x q \\ &\psi_{xx} - \tilde{\lambda}\psi_{yy} = \tilde{\alpha}(|q|^2)_x, \end{aligned} \tag{5}$$

belongs to the class of equations considered by Shulman in [2], as a particular case as mentioned above. Moreover, integrable cases of the DS system (5) were entirely classified with regard to its parameters.

Let us first mention the relation between (3) and (5). Their parameters are connected by

$$\tilde{\chi} = \chi - \frac{b}{\lambda}, \qquad \tilde{\lambda} = \lambda(1+\alpha\beta), \qquad \tilde{\alpha} = \alpha^2 + \frac{1}{\lambda}, \tag{6}$$

whilst $\tilde{\gamma} = \gamma$ and $\tilde{b} = b$. Finally we define $\psi$ by

$$\psi_x = \alpha\varphi_{1,x} + \varphi_{2,y} + \frac{1}{\lambda}|u|^2.$$

One can straightforwardly track down the fact that unless a certain relation among the parameters of the GDS system (3) is satisfied, defining the above transformations is impossible. This relation is given by

$$\alpha = (\alpha\lambda - \beta)(1+\alpha\beta), \tag{7}$$

which exists also in the papers mentioned at the beginning of this section, but in different forms.

Let us now note when the DS equation (5) is integrable. An observation is that in (5) the parameter $\tilde{\alpha}$ can easily be scaled by the map $\psi \mapsto \tilde{\alpha}\psi$, which only redefines the $\tilde{b}$ as $\tilde{b}\,\tilde{\alpha}$. It is essential that there remain just four parameters which fully determine the behavior of the DS equation.

With respect to these four parameters, there are only two integrable choices to be found in the literature; see for instance [1]:

$$(\tilde{\gamma}, \tilde{\chi}, \tilde{b}\,\tilde{\alpha}, \tilde{\lambda}) = (-1, 1, -2, -1) \quad \text{DS I}, \tag{8}$$

$$(\tilde{\gamma}, \tilde{\chi}, \tilde{b}\,\tilde{\alpha}, \tilde{\lambda}) = (1, -1, 2, 1) \quad \text{DS II}, \tag{9}$$

up to certain scalings. We immediately observe that DS I equation (8) is hyperbolic–elliptic type, whereas DS II equation (9) is elliptic–hyperbolic type and hence is the appropriate reduction target for our elliptic–hyperbolic–hyperbolic type GDS system (3).

In order to characterize the most general integrable cases of the elliptic–hyperbolic DS system (5), we scale the variables $x$ and $y$ in a way that $\tilde{\gamma}$ and $\tilde{\lambda}$ are normalized to 1. Then, we can scale $\psi$ so that no parameter remains in the second equation of (5). Consequently, the new form of the DS system given in (5) becomes

$$\begin{aligned} &iq_t + q_{xx} + q_{yy} = \tilde{\chi}|q|^2 q + \frac{\tilde{b}\,\tilde{\alpha}}{\tilde{\gamma}}\psi_x q \\ &\psi_{xx} - \psi_{yy} = (|q|^2)_x. \end{aligned}$$

Straightforward computations show that the system above can be transformed to DS II if and only if

$$2\tilde{\chi} + \frac{\tilde{b}\,\tilde{\alpha}}{\tilde{\gamma}} = 0, \tag{10}$$

by simple scalings of $q$ and $\psi$. Note that if $\tilde{\chi}$ is positive, then $q$ must be scaled by a purely imaginary constant to obtain the generic form of DS II given above by (9).

Because integrable cases of DS equation (5) are known, we are able to examine the compatibility of the necessary conditions to be derived in the next section with the condition (7) under which the GDS system (3) reduces to DS equation (5).

## 3. The vertex method and the GDS system

In this section we derive the first set of necessary conditions for integrability. In fact, vanishing of each vertex is required for integrability. However there are infinitely many vertices and hence it is not practical to check all these vertices vanish identically on the resonant surfaces. The underlying power of the method is either to receive nonvanishing lower order vertex and thereby conclude nonintegrability or for a certain parameter regime, for which the first vertex vanishes, to find a transformation which maps the system to an integrable system and hence conclude integrability. In [2] this method was used in both of these directions. It is worth noting that even the second vertex is computationally difficult to check for vanishing.

Following the standard techniques described and implemented in [3,2,16], the so-called first vertex vanishing equations were derived in [17]. The only further complexity is that the GDS system has three equations unlike the coupled Schrödinger equations considered by Zakharov and Shulman [16] or the family of equations treated by Shulman [2].

As described in [3] for cubic nonlinearities associated kinetic equation describes a four wave equation:

$$\omega(\mathbf{k}) + \omega(\mathbf{k}_1) = \omega(\mathbf{k}_2) + \omega(\mathbf{k}_3), \quad \mathbf{k} + \mathbf{k}_1 = \mathbf{k}_2 + \mathbf{k}_3, \tag{11}$$

where $\omega(\mathbf{k}) = \gamma(\xi_1)^2 + (\xi_2)^2$ is the dispersion law, $\mathbf{k} = (\xi_1, \xi_2)$, $\mathbf{k}_j = (\xi_1^{(j)}, \xi_2^{(j)})$ for $j = 1, 2, 3$ being the Fourier variables. The surface described by Eqs. (11) is called the (first) resonance surface and it is of co-dimension 3 in $\mathbb{R}^8$. We first equip the resonance surface with a rational parametrization following the recipe in [2]:

$$\begin{aligned} \mathbf{k} &= \left(P_1 + \frac{\kappa}{2\sqrt{\gamma}}(1-\tau\mu), P_2 + \frac{\kappa}{2}(\tau+\mu)\right), \\ \mathbf{k}_1 &= \left(P_1 - \frac{\kappa}{2\sqrt{\gamma}}(1-\tau\mu), P_2 - \frac{\kappa}{2}(\tau+\mu)\right), \\ \mathbf{k}_2 &= \left(P_1 + \frac{\kappa}{2\sqrt{\gamma}}(1+\tau\mu), P_2 + \frac{\kappa}{2}(\tau-\mu)\right), \\ \mathbf{k}_3 &= \left(P_1 - \frac{\kappa}{2\sqrt{\gamma}}(1+\tau\mu), P_2 - \frac{\kappa}{2}(\tau-\mu)\right), \end{aligned}$$

where $P_1, P_2, \kappa, \mu$ and $\tau$ are real parameters.

Using the parametrization above, first we check that dispersion law of the GDS system is nondegenerate with respect to the process (11). This means that every solution to the functional equation,

$$f(\mathbf{k}) + f(\mathbf{k}_1) = f(\mathbf{k}_2) + f(\mathbf{k}_3),$$

is of the form $f(\mathbf{k}) = A\omega(\mathbf{k}) + \mathbf{B}\cdot\mathbf{k} + C$ where $A, C$ are constants and $\mathbf{B}$ is a constant 2-vector. But this nondegeneracy is already verified in [2] and equally valid here since we have the same dispersion law and associated kinetic equation.

We next note that the GDS system (3) has a Hamiltonian:

$$H(u) = \int_{\mathbb{R}^2} \gamma|u_x|^2 + |u_y|^2 + \frac{\chi}{2}|u|^4 + \frac{b}{2}(\alpha\varphi_{1,x}^2 + \varphi_{2,y}^2)\, dxdy.$$

We now pass to Fourier variables $\mathbf{k}$ and using the last two equations in the GDS system express $\widehat{\varphi_1}$ and $\widehat{\varphi_2}$, the Fourier transforms of $\varphi_1$ and $\varphi_2$, respectively, in terms of $\mathbf{k}$ and the Fourier transform of $|u|^2$:

$$\begin{aligned} \widehat{\varphi_1}(\mathbf{k}) &= -i\frac{\alpha\xi_1^3 + (\beta - \lambda\alpha)\xi_1\xi_2^2}{\xi_1^4 - (1+\beta^2+\lambda)\xi_1^1\xi_2^2 + \lambda\xi_2^4}\widehat{|u|^2}(\mathbf{k}), \\ \widehat{\varphi_2}(\mathbf{k}) &= -i\frac{(\alpha\beta+1)\xi_1^2\xi_2 - \xi_2^3}{\xi_1^4 - (1+\beta^2+\lambda)\xi_1^1\xi_2^2 + \lambda\xi_2^4}\widehat{|u|^2}(\mathbf{k}). \end{aligned}$$

These two readily help us write the Fourier transform of $\alpha\varphi_{1,x}+\varphi_{2,y}$:

$$(\alpha\varphi_{1,x}+\varphi_{2,y})\widehat{\ }(\mathbf{k}) = \frac{\alpha^2\xi_1^4+(1+2\alpha\beta-\lambda\alpha^2)\xi_1^2\xi_2^2-\xi_2^4}{\xi_1^4-(1+\beta^2+\lambda)\xi_1^2\xi_2^2+\lambda\xi_2^4}\widehat{|u|^2}(\mathbf{k}) =: V(\mathbf{k})\widehat{|u|^2}(\mathbf{k}).$$

Transforming the first equation of (3) into the Fourier domain convolutions show up by usual properties of the transform:

$$\widehat{|u|^2u}(\mathbf{k}) = (\widehat{|u|^2}*\widehat{u})(\mathbf{k}),$$
$$((\alpha\varphi_{1,x}+\varphi_{2,y})u)\widehat{\ }(\mathbf{k}) = (V\widehat{|u|^2}*\widehat{u})(\mathbf{k}).$$

We wish to express the Hamiltonian in the Fourier domain in a form which is symmetric with respect to $\mathbf{k}\leftrightarrow\mathbf{k}_1$ and $\mathbf{k}_2\leftrightarrow\mathbf{k}_3$ because the process we are considering, that is (11), remains unchanged under these transformations. In order to accomplish this we symmetrize $V(\mathbf{k})$ in the convolutions by considering

$$\frac{1}{2}(V(\mathbf{k}-\mathbf{k}_2)+V(\mathbf{k}-\mathbf{k}_3))$$

instead of using just $V(\mathbf{k}-\mathbf{k}_2)$ or $V(\mathbf{k}-\mathbf{k}_3)$. Consequently, we obtain the Hamiltonian as

$$H(u) = \int_{\mathbb{R}^2}(\gamma\xi_1^2+\xi_2^2)\widehat{u}(\mathbf{k})\overline{\widehat{u}}(\mathbf{k})\,d\mathbf{k} + \frac{1}{2}\int_{\mathbb{R}^8}T(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)\overline{\widehat{u}}(\mathbf{k})\overline{\widehat{u}}(\mathbf{k}_1)\widehat{u}(\mathbf{k}_2)\widehat{u}(\mathbf{k}_3) \times\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\,d\mathbf{k}d\mathbf{k}_1d\mathbf{k}_2d\mathbf{k}_3,$$

where the overbar denotes complex conjugation, $\delta(\cdot)$ the Dirac-delta distribution and:

$$T(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) = \chi+\frac{b}{2}(V(\mathbf{k}-\mathbf{k}_2)+V(\mathbf{k}-\mathbf{k}_3)). \quad (12)$$

We also note that in the derivation of $H(u)$ we have employed the fact that the dispersion law $\omega$ and $V$ are even functions of $\xi_1$ and $\xi_2$.

The $T$ defined in (12) is called the first vertex and required to vanish identically on the resonance surface parametrized by $P_1, P_2, \kappa, \mu$ and $\tau$, for the GDS system to be integrable by means of the inverse scattering transform. Noting that

$$\mathbf{k}-\mathbf{k}_2 = \left(-\frac{\kappa\tau\mu}{\sqrt{\gamma}},\kappa\mu\right), \qquad \mathbf{k}-\mathbf{k}_3 = \left(\frac{\kappa}{\sqrt{\gamma}},\kappa\tau\right),$$

and substituting these in (12) we express the first vertex $T$ on the resonance surface. One straightforwardly observes that in the expression of $T$ on the resonance surface, only the parameter $\tau$ exists. This means that the first vertex restricted to this surface is a rational curve parametrized by $\tau$. Requiring the vanishing of $T$ restricted to the resonance surface, we receive

$$\begin{aligned}0 = {} & \tau^8[(2\chi\lambda+b\lambda\alpha^2-b)\gamma^2]+\tau^6[-2\chi(\beta^2\\ & +\lambda+1)(\lambda\gamma^2+1)\gamma+b(2\alpha\beta+1-\alpha^2\lambda)(\lambda\gamma^2+1)\gamma\\ & -b(\beta^2+\lambda+1)(\alpha^2-\gamma^2)\gamma]\\ & +\tau^4[2\chi(1+(\beta^2+\lambda+1)^2\gamma^2+\lambda^2\gamma^4)\\ & +2b(\alpha^2-\lambda\gamma^4-(2\alpha\beta+1-\alpha^2\lambda)(\beta^2+\lambda+1)\gamma^2)]\\ & +\tau^2[-2\chi(\beta^2+\lambda+1)(\lambda\gamma^2+1)\gamma\\ & +b(2\alpha\beta+1-\alpha^2\lambda)(\lambda\gamma^2+1)\gamma\\ & -b(\beta^2+\lambda+1)(\alpha^2-\gamma^2)\gamma]+(2\chi\lambda+b\lambda\alpha^2-b)\gamma^2\end{aligned} \quad (13)$$

which must hold true identically for each $\tau$. Because of $\tau\leftrightarrow 1/\tau$ duality of the parametric representation in the vertex method, we expectedly have the same coefficients for two $\tau$-terms whose powers add up to 8. Therefore, as visible from (13), three linearly independent equations among six parameters showing up in the GDS system (3) must be satisfied.

Starting from the coefficient of $\tau^8$,

$$2\chi = \frac{b}{\lambda}(1-\lambda\alpha^2) \quad \text{(NC 1)}$$

must be satisfied. Using this in the coefficient of $\tau^6$, we get

$$2\alpha\beta-\alpha^2\lambda+1 = (\beta^2+\lambda+1)\frac{1-\lambda^2\gamma^2\alpha^2}{\lambda(1+\lambda\gamma^2)} \quad \text{(NC 2)}$$

as the second condition for the vanishing of the first vertex. We now use (NC 1) and (NC 2) in the coefficient of $\tau^4$ whence the final equation to hold turns out to be

$$(1+\lambda\alpha^2)(1-\lambda^2\gamma^4)\left[1-\left(\frac{\gamma(\beta^2+\lambda+1)}{1+\lambda\gamma^2}\right)^2\right] = 0.$$

Using the assumed parameter structure saying $\gamma,\lambda > 0$, this product vanishing yields only either of the following two conditions:

$$\lambda\gamma^2 = \gamma(\beta^2+\lambda+1)-1, \quad \text{(NC 3a)}$$
$$\lambda\gamma^2 = 1. \quad \text{(NC 3b)}$$

**Lemma 2** (*Necessary Conditions for Integrability*)**.** *In order that the GDS system (3) is integrable, the necessary conditions are (NC 1) and (NC 2), and either (NC 3a) or (NC 3b).*

It is important to note that these are not sufficient conditions. Moreover, they are not the only necessary conditions. Vanishing of all vertices, which are infinitely many, is necessary for an equation to be integrable. To conclude integrability, the equation subject to a parameter regime constrained by means of necessary conditions could be transformed to an equation which is already known to be integrable, if this is possible. Indeed, Shulman implemented this strategy in [2] to justify that the only integrable equations in the family he considered are DS I and DS II equations.

## 4. Problem of integrability

In this section we examine the GDS system in view of Lemma 2. This is to say, we separate the necessary conditions into cases with regard to the conditions obtained by the vertex method and analyze them in order. In each of the cases (NC 1) and (NC 2) must be satisfied. Hence the cases we will consider below are classified with respect to the third set of conditions:

Both (NC 3a) and (NC 3b) hold: Case I.
(NC 3a) holds but (NC 3b) does not: Case II.
(NC 3b) holds but (NC 3a) does not: Case III.

In treatment of the first two cases we do not need to consider the more restricted GDS system (1) and draw conclusions directly for the elliptic–hyperbolic–hyperbolic GDS system (3), whereas the third case could only be resolved by considering the system (1) which comprises more restrictive relations among parameters. The reason for the third case being harder than the first two is that the vertex vanishing condition (NC 3a) makes the reduction to a DS system possible and hence what remains is to check whether this is an integrable reduction or not. On the other hand, in the last case no such reduction comes granted and even when the system is externally forced to reduce to a DS system it turns out to be a nonintegrable reduction. We now give the detailed analysis of each case described above.

### 4.1. Case I

This is clearly the simplest case since both of the conditions (NC 3a) and (NC 3b) hold. Note that just one of them is necessary for integrability, but this case deals with the extreme possibility. That is

$$\lambda\gamma^2 = 1 = \gamma(\beta^2+\lambda+1) - 1.$$

But then putting $\lambda = 1/\gamma^2$ we get a quadratic equation for $\gamma > 0$:

$$\gamma^2(\beta^2+1) - 2\gamma + 1 = 0,$$

which is equivalent to $(\gamma-1)^2 + \gamma^2\beta^2 = 0$. Hence we receive $\gamma = 1$ and $\beta = 0$ as the only solutions, which in turn entail $\lambda = 1$.

Thus, if both necessary conditions (NC 3a) and (NC 3b) are assumed to hold, the GDS system (3) is integrable only if $\gamma = \lambda = 1$ and $\beta = 0$. In addition (NC 2) is identically satisfied and (NC 1) takes the form $2\chi = b(1-\alpha^2)$ which is now the only nontrivial necessary condition for integrability of GDS (3).

On the other hand, if we check what happens to the reducibility constraint (7), we see that it has become a tautology. What remains to show is that this reducible GDS system can also achieve a DS II reduction rather than a mere, perhaps nonintegrable, DS reduction. In order to check this, we substitute the data we have obtained so far into the constraints (6) and write the integrability condition (10) of the elliptic–hyperbolic DS system in terms of the parameters of the GDS system (3). This yields

$$2\chi + b(\alpha^2 - 1) = 0,$$

which coincides with the first vertex vanishing condition (NC 1). Hence, the reduction is necessarily integrable.

**Lemma 3** (*Resolution of Case I*). *Suppose (NC 3a) and (NC 3b) hold true. Then the GDS system* (3) *reduces to a DS system and the necessary condition (NC 2) is identically satisfied. Moreover, the necessary condition (NC 1) coincides with integrability of the resulting DS system.*

### 4.2. Case II

We now assume $\gamma(\beta^2+\lambda+1) = 1+\lambda\gamma^2$ and $\lambda\gamma^2 \neq 1$. Here $\lambda, \gamma > 0$. Solving the first relation for $\lambda$, we get

$$\lambda = \frac{\gamma(\beta^2+1)-1}{\gamma^2-\gamma} \tag{14}$$

where the denominator is well defined since $\gamma > 0$ and $\gamma = 1$ is not possible as it yields $\beta = 0$ from (NC 3a) and consequently $\lambda = 1$ by (NC 2), which contradicts the assumption $\lambda\gamma^2 \neq 1$. So we set $\gamma \neq 1$.

Now substituting $\lambda$ from (14) into the necessary condition (NC 2) and simplifying, we receive $(\gamma(\beta+\alpha(\beta^2+1)) - \alpha)^2 = 0$. The only solution is

$$\gamma = \frac{\alpha}{\beta+\alpha(\beta^2+1)} \tag{15}$$

provided that the denominator is not zero. But this is fine for otherwise $\alpha = 0$ by (NC 2) which in turn entails $\beta = 0$. Vanishing of $\beta$ makes $\lambda\gamma = 1$ by virtue of (14). Using the remaining equation (NC 1) to annihilate the first vertex, we get $2\chi = b/\lambda$, where $\lambda \neq 1$ since else implies $\gamma = 1$ from $\lambda\gamma = 1$.

Furthermore, $\alpha = \beta = 0$ guarantees that the reducibility condition $\alpha = (\alpha\lambda - \beta)(1+\alpha\beta)$ holds true identically. This reduction is integrable only when (10) is true, which becomes

$$2\chi + b\left(1 - \frac{1}{\lambda}\right) = 0.$$

Rewriting this by putting $b/\lambda$ in the place of $2\chi$, we end up with two possibilities: either $b = 0$ or $\lambda = 1$. The latter is immediately impossible as described above, whilst the former entails $\chi = 0$ and hence the system totally decouples and becomes linear. Hence we set $\beta + \alpha(\beta^2+1) \neq 0$.

We now express $\lambda$ in terms of $\alpha$ and $\beta$ by eliminating $\gamma$ in (14) via (15)

$$\lambda = \frac{\beta+\alpha(\beta^2+1)}{\alpha(1+\alpha\beta)}. \tag{16}$$

The expressions (15) and (16) determine the parameters $\lambda$ and $\gamma$ purely in terms of $\alpha$ and $\beta$. It is obvious to check that with $\lambda$ in (16) the reducibility condition (7) is tautologically satisfied, meaning that if the necessary conditions (NC 2) and (NC 3a) for integrability hold (and $\gamma \neq 1$ and $\beta+\alpha(\beta^2+1) \neq 0$) then the GDS system is reducible to a DS equation. We now have to ask whether this is an integrable reduction.

We use (15) and (16) in the reducibility relations (6) and express the parameters of the DS system in terms of $\alpha$ and $\beta$. These expressions together with the restriction arising from the vertex vanishing equation (NC 1) simplify the integrability condition (10) for the DS system to

$$\frac{\beta(1+\alpha\beta)^3}{\beta+\alpha(\beta^2+1)} = 0.$$

This leaves us with two options: either $\beta = 0$ or $1+\alpha\beta = 0$.

If $\beta = 0$ then (NC 3a) reduces to $\lambda\gamma = 1$ and (NC 2) to $\alpha^2(\lambda-1) = 0$. But $\alpha = 0$ is impossible because then $\beta+\alpha(\beta^2+1)$ becomes zero. However, $\lambda = 1$ is not possible either as it would imply $\gamma = 1$ from $\lambda\gamma = 1$.

If $1+\alpha\beta = 0$ then from the reducibility constraint (7), which holds identically in this case, we get $\alpha = 0$. This contradicts the assumption $1+\alpha\beta = 0$.

The following lemma summarizes our findings.

**Lemma 4** (*Resolution of Case II*). *Suppose (NC 3a) together with the necessary conditions (NC 1) and (NC 2) are satisfied but (NC 3b) is not. Then the GDS system* (3) *reduces to a DS system. However, this reduction cannot achieve the DS II system and hence is not integrable. Therefore, there is no integrable system in the family* (3) *under the above hypotheses.*

### 4.3. Case III

As the last case we let the necessary condition (NC 3b) be satisfied but (NC 3a) not hold true. That is $\lambda\gamma^2 = 1$ but $\gamma(\beta^2+\lambda+1) \neq 1+\lambda\gamma^2$.

Substituting $\gamma = 1/\sqrt{\lambda}$, recalling both are positive in our realm, the second necessary condition's negation becomes $\beta^2 + \left(\sqrt{\lambda}-1\right)^2 \neq 0$. Hence $\beta \neq 0$ or $\lambda \neq 1$. The vertex vanishing condition (NC 1) does not have an immediate simplification, however (NC 2) becomes

$$\alpha^2\lambda^2 - [1+4\alpha\beta+\alpha^2(1+\beta^2)]\lambda + 1 + \beta^2 = 0. \tag{17}$$

This quadratic equation for $\lambda$ has two positive solutions (real roots are necessarily of the same sign) if $B := 1+4\alpha\beta+\alpha^2(1+\beta^2)$ satisfies two conditions:

$$B > 0 \quad \text{and} \quad B^2 - 4\alpha^2(1+\beta^2) \geq 0.$$

It is not obvious whether these two conditions are compatible or not but there does not seem to exist any a priori obstruction. It is basically because the restrictions among the parameters are not strong enough to let us draw a nonintegrability type conclusion. At this stage we can take two different paths: either checking if the GDS system subject to the mentioned parameter regime has got an integrable reduction to the DS equation or considering the genuinely physical system (1) in which case we need to get back to the original physical parameters and employ the relations among them.

When we investigate if the GDS system has an integrable reduction to the DS equation, keeping in mind that our reduction scheme may not be unique, the answer is negative. We examine this below in three cases by assuming that the reduction is possible, namely $\alpha = (\alpha\lambda - \beta)(1 + \alpha\beta)$.

If $\beta \neq 0$ and $\lambda \neq 1$, then $\gamma = 1$. We must set $\alpha \neq 0$ for otherwise the reducibility condition (7) yields $\beta = 0$. But then the vertex vanishing condition (17) becomes

$$\frac{\beta}{\alpha(1+\alpha\beta)^2}((1+\alpha\beta)^2 + \alpha^2) = 0$$

which cannot be true. Note here also that in order that (7) holds true, $1 + \alpha\beta \neq 0$ should be the case, for otherwise we get $\alpha = 0$ leading to a contradiction.

If $\beta = 0$ and $\lambda \neq 1$, then the reducibility condition (7) gives either $\alpha = 0$ or $\lambda = 1$, where the latter is impossible. Letting $\alpha = 0$ in (17) we once again receive $\lambda = 1$. So this choice is not possible either.

Lastly, we assume $\beta \neq 0$ and $\lambda = 1$ implying $\gamma = 1$. Under these suppositions, reducibility constraint simplifies to

$$\beta = \frac{\alpha^2 - 1}{\alpha}, \quad \alpha \neq 0.$$

On the other hand, (17) is equivalent to

$$\beta = \frac{4\alpha}{1 - \alpha^2}, \quad |\alpha| \neq 1,$$

noticing that $|\alpha| = 1$ produces an inconsistency in (17). But, evidently the two expressions above for $\beta$ cannot be true simultaneously.

This shows that vanishing conditions for the first vertex associated with the GDS system (3) are not compatible with the reduction scheme we have described, independent of whether the reduction is integrable or not. It is clear that this incompatibility result is not satisfactory enough. So we turn our attention to the original GDS system (1) and take the physical constraints into account.

It is obvious that passing to physical parameters makes the system seem to be more complicated. However, in this representation, we gain a more detailed information on the coefficients essentially owing to the fact in [6]:

$$c_g = \frac{c_2^2 k}{\omega}(1 + 8m^2k^2).$$

Using the above identity together with $\omega = c_2 k(1 + 4m^2k^2)^{1/2}$ and setting $s = 4m^2k^2 > 0$ in what follows, we immediately get

$$c_g = \frac{1+2s}{\sqrt{1+s}} c_2.$$

Similarly we rewrite $p$ and $r$ given in (2) as

$$p = \frac{s(3+2s)}{2(1+s)^{3/2}} \frac{c_2}{k}, \qquad r = \frac{1+2s}{2\sqrt{1+s}} \frac{c_2}{k},$$

which we combine with expressions of $a_1$ and $a_2$ in terms of $c_1$ and $c_2$ and receive the constants $\gamma$, $\alpha$, $\beta$ and $\lambda$ as functions of $c_1$, $c_2$ and $s$ (with the exception that $\alpha$ still contains $\gamma_3/\gamma_1$).

Particularly important are

$$\gamma = \frac{s(3+2s)}{(1+2s)(c_2^2(1+2s)^2 - c_1^2(1+s))} c_2^2$$

$$\lambda = \frac{c_2^2(1+2s)^2 - c_1^2(1+s)}{s(3+4s)} \frac{c_1^2}{c_2^4}$$

which when substituted in the assumed necessary condition $\lambda\gamma^2 = 1$ yield the positive constant $c_1$ in terms of $c_2$ as

$$c_1 = \frac{\sqrt{3+4s}(1+2s)^2}{\sqrt{3 + 28s + 56s^2 + 48s^3 + 16s^4}} c_2.$$

This relation between $c_1$ and $c_2$ must hold in order that the GDS system is integrable. On the other hand, it simplifies the expressions of $\gamma$, $\alpha$, $\beta$ and $\lambda$ to a dependency upon $s$ (for $\alpha$ the $\gamma_3/\gamma_1$ dependency survives).

We give the latter three of these parameters in terms of $s$:

$$\alpha = \frac{\gamma_3}{\gamma_1} \frac{\sqrt{3+4s}\sqrt{1+s}}{\sqrt{s}} \frac{3 + 28s + 56s^2 + 48s^3 + 16s^4}{(1+2s)^2(3+2s)^2}$$

$$\beta = 16 \frac{s^{3/2}(1+s)^{3/2}(3 + 6s + 4s^2)}{3 + 28s + 56s^2 + 48s^3 + 16s^4}$$

$$\lambda = \frac{(3+2s)^2(1+2s)^6}{3 + 28s + 56s^2 + 48s^3 + 16s^4}$$

and use them in the Eq. (17), which is another necessary condition for integrability. Then using the Maple software, we solve the resulting quadratic equation for $\gamma_3/\gamma_1$ as a function of $s$.

Recall that the physical constant $\frac{\gamma_3}{\gamma_1} > 1$. But the procedure above calculates this quotient as less than $1/5$ for all values of $s$. This proves that in the assumed parameter regime the GDS system (1) is not integrable.

**Lemma 5** (*Resolution of Case III*)**.** *Assume that (NC 3b) together with the necessary conditions (NC 1) and (NC 2) hold true but (NC 3a) does not. Then the parameters in the GDS system* (1) *cannot be chosen consistently so as to satisfy the constraints dictated by the derivation. In other words, the system* (1)*, with the restrictions on its parameters, is not integrable.*

Combining Lemmas 3–5, the main result of this paper follows.

## 5. Discussion and concluding remarks

In this paper, we have considered the generalized Davey–Stewartson system in the elliptic–hyperbolic–hyperbolic category which is particularly important because certain physical cases of this system belong to this class. Our approach is based upon the method developed by Zakharov and Shulman, the so-called vertex method. This method may be interpreted as a test for an equation to be integrable by means of the inverse scattering transform.

The vertex method applied successfully to various classes of equations by Zakharov and Shulman is essentially a test for nonintegrability. The reason for this is clear: the method produces necessary conditions. In fact, concluding integrability is impossible without utilizing other approaches, as done in [2,16]. The method is quite algebraic and computational rather than being functional analytic. Although we have employed only the necessity of vanishing of the first vertex, we must note that vanishing of each vertex is obligatory for an equation to be integrable. This idea may be used if the first vertex vanishing does not provide sufficient information. However, higher order vertices are computationally difficult to obtain and complicated enough to extract information from their vanishing on the relevant resonance surfaces.

We remark that in cases I and II we have nonintegrability result for the elliptic–hyperbolic–hyperbolic GDS system (3), whilst in case III our nonintegrability result works only for the physical GDS system (1) which is a subclass of (3). Thus, our main result applies to this physical version only. Yet, in two cases of the three, nonexistence of a new integrable system for the more general class (3) is shown.

An algebraically challenging problem appears when the coefficients in the GDS system are allowed to assume any real value.

This may include cases such as elliptic–elliptic–elliptic, hyperbolic–elliptic–elliptic or elliptic–elliptic–hyperbolic classes for instance. In some of these situations an analysis in the spirit of this study does not resolve the integrability issue and hence the hope for finding a new integrable system in $2+1$ dimensions survives. Another point to be made for the most general case is that the integrable systems (4) and (28) of [2] might become important in the sense that reducibility to these equations should be investigated as we have similarly profited from reducibility to the DS II system in this work.

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