Please refresh/test your knowledge/interest in this course.

Math 352 – Homework 0 (no grading)

1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be differentiable at each $(x, y) \in \mathbb{R}^2$. Let $g_1(x, y, z) = x^2 + y^2 + z^2$ and $g_2(x, y, z) = x + y + z$, and let $\boldsymbol{g} = (g_1, g_2)$. Define $h = f \circ \boldsymbol{g}$. Show that:

$$\|\nabla h\|^2 = 4(D_1 f)^2 g_1 + 4(D_1 f)(D_2 f) g_2 + 3(D_2 f)^2,$$

where $D_j f$ means the partial derivative of f with respect to its jth argument.

2. Let $F(x+y+z, Ax+By) = 0$ for some differentiable F with $\nabla F \neq \mathbf{0}$, and $A, B \in \mathbb{R}\backslash\{0\}$. This level set naturally defines z as a function of x and y. Prove that:

$$A\frac{\partial z}{\partial y} - B\frac{\partial z}{\partial x} = \text{constant}.$$

3. Let y_1 and y_2 be two solutions of the DE $y'' + p(t)y' + q(t)y = 0$, where $p(t)$ and $q(t)$ are continuous functions of t on an open interval I.

(a) Prove that if y_1 and y_2 are zero at the same point in I then they cannot form a fundamental set of solutions.

(b) Prove that if y_1 and y_2 have maxima or minima at the same point in I then they cannot form a fundamental set of solutions.

(c) Prove that if y_1 and y_2 have a common point of inflection $t_0 \in I$ then they cannot be fundamental solutions unless p and q are zero at t_0.

4. Consider the boundary problem on $0 \leq x \leq l$:

$$u''(x) + u'(x) = f(x)$$
$$u(0) = u'(0) = \tfrac{1}{2}[u'(l) + u(l)]$$

where f is a given arbitrary function.

(a) Is the solution unique? Explain.

(b) Does a solution necessarily exist or is there a condition that f must satisfy for existence? Explain.

5. Solve the boundary problem $u'' = 0$ for $0 < x < 1$ with $u'(0) + ku(0) = 0$ and $u'(1) \pm ku(1) = 0$. Do the $+$ and $-$ cases separately. What is special about $k = 2$?

6. If $\boldsymbol{f} : \mathbb{R}^3 \to \mathbb{R}^3$ is continuous and $\|\boldsymbol{f}(\boldsymbol{x})\| \leq \dfrac{1}{\|\boldsymbol{x}\|^3 + 1}$ for all $\boldsymbol{x} \in \mathbb{R}^3$, show that:

$$\iiint_{\mathbb{R}^3} \boldsymbol{\nabla} \cdot \boldsymbol{f}\, dV = 0.$$

Take D to be a large ball, apply Gauss's divergence theorem and let the radius of the ball tend to infinity.

7. If Γ is a closed anticlockwise directed simple curve in $\mathbb{R}^2$ with interior Ω and $\boldsymbol{F}$ is a smooth vector field on an open set containing $\Omega \cup \Gamma$, show that Green's theorem is equivalent to:

$$\iint_\Omega \operatorname{div} \boldsymbol{F}\, dA = \int_\Gamma \boldsymbol{F} \cdot \boldsymbol{n}\, ds,$$

where $\boldsymbol{n}$ is the outward unit normal to Γ.

8. Let $\boldsymbol{n}$, Γ and Ω be as in problem 7. If u is smooth and harmonic on Ω, i.e., $u_{xx}+u_{yy}=0$ and continuous on $\Omega \cup \Gamma$, show that:

$$\int_\Gamma \nabla f \cdot \boldsymbol{n}\, ds = 0.$$

9. If Ω is open in $\mathbb{R}^2$ and $f : \Omega \to \mathbb{R}$ is smooth, show that:

$$\text{Area}\,(\text{graph}\, f) = \iint_\Omega (1 + \|\nabla f\|^2)^{1/2}\, dA.$$

10. Use Stokes's theorem to evaluate the surface integral:

$$\iint_\Sigma \text{curl}\,(y^2, xy, xz)\, dS$$

where Σ is the hemi-sphere $x^2 + y^2 + z^2 = 1,\ z > 0$, oriented so that the unit normal has positive z coordinate.