MODEL THEORY OF FIELDS WITH MULTIPLICATIVE GROUPS

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B.A., Istanbul Bilgi University, 2001

DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics
in the Graduate College of the
University of Illinois at Urbana-Champaign, 2008

Urbana, Illinois

Abstract

The general theme of this thesis is the model theory of fields expanded by multiplicative groups. We first consider a real closed field expanded by two subgroups of the positive part of that field. Then we investigate an algebraically closed field with a multiplicative subgroup satisfying a certain diophantine condition with respect to a subfield of the algebraically closed field. We also study the real field with a distinguished subgroup of the unit circle group in the complex plane.

To Ali Ihsan and Meral Günaydın

Acknowledgments

I am indebted to my advisor Lou van den Dries for his kind support and patience. Without his guidance this thesis would not exist. It took five years, but now I know that “conditions are satisfied” and “things have properties”.

I would like to thank my family for their life-long affection and encouragement. I also owe many thanks to my friends for not letting this thesis take over my life. For fear of forgetting someone I am not going to spell out your names. You know who you are.

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Chapter 1

Introduction

In this thesis we study fields with one or more multiplicative subgroups satisfying certain conditions. A first model-theoretic study of one such structure is in [7]. There the elementary theory of $(\mathbb{R}, 2^{\mathbb{Z}})$ was determined, where $\mathbb{R}$ is the real field and $2^{\mathbb{Z}}$ is the multiplicative group of powers of 2, and the definable relations were shown to be boolean combinations of existentially definable relations. In that paper the main tool is elementary valuation theory; this tool, however, does not apply to groups like $2^{\mathbb{Q}} \subseteq \mathbb{R}^{\times}$ and $2^{\mathbb{Z}}3^{\mathbb{Z}} \subseteq \mathbb{R}^{\times}$. A new approach is due to Boris Zilber in [26] and [27]; in the former of these papers Zilber axiomatizes $(\mathbb{C}, \mathbb{U})$, where $\mathbb{C}$ is the field of complex numbers and $\mathbb{U}$ is the multiplicative group of complex roots of unity and in the latter he proves that in $(\mathbb{R}, \mathbb{U})$ all the definable relations are boolean combinations of existentially definable relations. In [26] the main tool is a theorem of Mann from [17] on sums of roots of unity. This led to [9], where results of the type above are obtained for structures $(\mathbb{R}, G)$, with G any multiplicative subgroup of $\mathbb{R}^{>0}$ of finite rank, for instance $2^{\mathbb{Q}}$ or $2^{\mathbb{Z}}3^{\mathbb{Z}}$. (A commutative group G is said to have *finite rank* if G has a finitely generated subgroup H such that G/H is a torsion group.)

An problem from [7] is to extend its results to $(\mathbb{R}, 2^{\mathbb{Z}}, 3^{\mathbb{Z}})$, and this problem remains open. As shown in a stronger form in the last example of Section 6.2 below, the sets $2^{\mathbb{Z}}$ and $3^{\mathbb{Z}}$ are not definable in $(\mathbb{R}, 2^{\mathbb{Z}}3^{\mathbb{Z}})$ and thus [9] does not by itself yield a solution to this problem. In Section 6.2 we extend the results from [7] and [9] to structures like $(\mathbb{R}, 2^{\mathbb{Z}}3^{\mathbb{Z}}, 2^{\mathbb{Z}})$, but there is no obvious reduction of the open problem of [7] to this situation: the set $3^{\mathbb{Z}}$ is not definable in $(\mathbb{R}, 2^{\mathbb{Z}}3^{\mathbb{Z}}, 2^{\mathbb{Z}})$; see Section 6.2.

A key fact about the (multiplicative) subgroups of $\mathbb{C}^{\times}$ and $\mathbb{R}^{>0}$ of finite rank is that they have the *Mann Property*, as it is called in [9], where this property is studied in an axiomatic setting. To define this property, let K be a field and G a subgroup of its multiplicative group $K^{\times}$. Consider an equation

$$a_1x_1 + \cdots + a_nx_n = 1 \qquad (*)$$

with $n \geq 1$ and *nonzero* coefficients $a_1, \ldots, a_n \in K$. A *solution* of $(*)$ in G is a tuple $(g_1, \ldots, g_n) \in G^n$ such that

$$a_1 g_1 + \cdots + a_n g_n = 1,$$

and a *nondegenerate* solution of $(*)$ in G is a solution $(g_1, \ldots, g_n)$ of $(*)$ in G such that $\sum_{i \in I} a_i g_i \neq 0$ for each nonempty subset I of $\{1, \ldots, n\}$. We say that G has the *Mann property* (in K) if for every nonzero $a_1, \ldots, a_n$ from the *prime field* of K, the equation $(*)$ has only finitely many nondegenerate solutions in G. We call it by this name because Mann [17] showed that $\mathbb{U}$ has the Mann property in $\mathbb{C}$. Some twenty years later came the proof of the deeper theorem that any multiplicative group of finite rank in any field of characteristic 0 has the Mann property; see [11], [16], [21]. Mann's proof, however, has the virtue of being effective: for any equation $(*)$ with nonzero rational a_i it constructs all nondegenerate solutions in $\mathbb{U}$.

In Chapter 3, we recall some facts from [9] and prove some related results. In the last section of that chapter we also give some examples illustrating that the main results of [9] do not go through when the real field $\mathbb{R}$ is replaced by its o-minimal expansion $\mathbb{R}_{\text{an}}$.

One objective of this thesis is to study structures (K, A, G), where K is a real closed field, A is a subgroup of $K^{>0}$ with the Mann property and A is dense in $K^{>0}$, and G is an ordered subgroup of A which either is dense in A or has a smallest element $\mathrm{b}(G)$ greater than 1 such that for every $a \in A$ there is $g \in G$ with $g \leq a < g\,\mathrm{b}(G)$. (Note that then also G has the Mann property.) We achieve this goal to some extent in Chapter 6, with preparations in Chapters 4 and 5. If A/G is finite, then A is definable in (K, G), and so this situation reduces to that of [9]. Hence we also assume that A/G is infinite.

For a multiplicatively written abelian group A and a positive integer d, define

$$A^{[d]} := \{a^d : a \in A\} \qquad \text{(its subgroup of } d^{\text{th}} \text{ powers)},$$

with the corresponding elementary invariant

$$[d]A := \begin{cases} |A/A^{[d]}| & \text{if } A/A^{[d]} \text{ is finite,} \\ \infty & \text{otherwise.} \end{cases}$$

Throughout this dissertation we let p range over the set of prime numbers.

Let $\mathcal{L}_{\text{o}} = \{0, 1, +, -, \cdot, <\}$ be the language of ordered rings, and let $\mathcal{L}_{\text{o}}(U, V)$ the language extending $\mathcal{L}_{\text{o}}$ by two distinct unary predicates U, V.

Let Δ be a dense subgroup of $\mathbb{R}^{>0}$ with the Mann property. For each equation

$$a_1 x_1 + \cdots + a_n x_n = 1 \qquad (n \geq 2,\ a_1, \ldots, a_n \in \mathbb{Q}^{\times})$$

choose a finite list of its nondegenerate solutions in Δ,

$$\delta_1 = (\delta_{11}, \ldots, \delta_{1n}), \ldots, \delta_k = (\delta_{k1}, \ldots, \delta_{kn}),$$

and let the corresponding *Mann axiom of* Δ be the sentence

$$\forall y \Big[\big(U(y) \wedge \sum_{i=1}^{n} a_i y_i = 1 \wedge \bigwedge_{I} \sum_{i \in I} a_i y_i \neq 0 \big) \longrightarrow \bigvee_{j=1}^{k} y = \delta_j \Big]$$

in the language $\mathcal{L}_o(U,V)$ with names for the elements of Δ, where $y = (y_1, \ldots, y_n)$ is a tuple of distinct variables, $U(y)$ abbreviates $U(y_1) \wedge \cdots \wedge U(y_n)$, the conjunction $\bigwedge_I$ is over all nonempty $I \subseteq \{1, \ldots, n\}$, "$\sum_{i=1}^{n} a_i y_i = 1$" and "$\sum_{i \in I} a_i y_i \neq 0$" represent certain obvious formulas in the language of rings, and $y = \delta_j$ abbreviates $y_1 = \delta_{j1} \wedge \cdots \wedge y_n = \delta_{jn}$.

Let Γ be an ordered subgroup of Δ such that Δ/Γ is infinite, and define the *Mann axioms* for Γ likewise. Construe $\big(\mathbb{R}, \Delta, \Gamma, (\delta)_{\delta \in \Delta}\big)$ as a structure for the language $\mathcal{L}_o(U,V)$ augmented by names for elements of Δ.

We first consider the case that Γ is dense in Δ. In that case we have the following axiomatisation of the theory of $\big(\mathbb{R}, \Delta, \Gamma, (\delta)_{\delta \in \Delta}\big)$ proved in Section 6.1 (see page 58).

Theorem 1.0.1. *Let K be a real closed ordered field, let A, G be dense subgroups of $K^{>0}$ such that $G \subseteq A$ and A/G is infinite, and let a group homomorphism $\delta \mapsto \delta' : \Delta \to A$ be given. Then $\big(K, A, G, (\delta')_{\delta \in \Delta}\big)$ is elementarily equivalent to $\big(\mathbb{R}, \Delta, \Gamma, (\delta)_{\delta \in \Delta}\big)$ if and only if*

(1) $\delta \in \Gamma \Leftrightarrow \delta' \in G$, *for all* $\delta \in \Delta$;

(2) *for each* $\delta \in \Delta$ *and each* p, *if* δ *is not a* p^{th} *power in* Δ, *then* δ' *is not a* p^{th} *power in* A ;

(3) *for each* $\gamma \in \Gamma$ *and each* p, *if* γ *is not a* p^{th} *power in* Γ, *then* γ' *is not a* p^{th} *power in* G ;

(4) $[p]A = [p]\Delta$ *and* $[p]G = [p]\Gamma$ *for each* p ;

(5) *for all* $a_1, \ldots, a_n \in \mathbb{Z}$ *and* $\delta_1, \ldots, \delta_n \in \Delta$,

$$a_1\delta_1 + \cdots + a_n\delta_n > 0 \Longleftrightarrow a_1\delta'_1 + \cdots + a_n\delta'_n > 0 \ ;$$

(6) $\big(K, A, (\delta')_{\delta \in \Delta}\big)$ *satisfies the Mann axioms of* Δ;

(7) $\big(K, G, (\gamma')_{\gamma \in \Gamma}\big)$ *satisfies the Mann axioms of* Γ.

Note that the "only if" part of the theorem is obvious. This theorem takes care of structures like $(\mathbb{R}, 2^{\mathbb{Q}}3^{\mathbb{Q}}, 2^{\mathbb{Z}}3^{\mathbb{Z}})$ and $(\mathbb{R}, 2^{\mathbb{Q}}3^{\mathbb{Z}}, 2^{\mathbb{Q}})$. To understand the second kind of triple, for example $(\mathbb{R}, 2^{\mathbb{Z}}3^{\mathbb{Z}}, 2^{\mathbb{Z}})$, let Δ be as above and Γ a subgroup of Δ with a smallest element larger than 1, say $\mathrm{b}(\Gamma)$. Then it is easy to see that $\Gamma = \mathrm{b}(\Gamma)^{\mathbb{Z}}$, so Γ is a regularly discrete ordered abelian group in the sense of [22]. The next theorem is proved in Section 6.2 (see page 62).

Theorem 1.0.2. *Let K be a real closed ordered field, A a dense subgroup of $K^{>0}$, and $\delta \mapsto \delta' : \Delta \to A$ a group homomorphism. Let G be a subgroup of A such that $\mathrm{b}(\Gamma)'$ is the smallest element of G larger than 1 and for every $a \in A$, there is $g \in G$ satisfying $g \le a < g\,\mathrm{b}(\Gamma)$. Then $\big(K, A, G, (\delta')_{\delta\in\Delta}\big) \equiv \big(\mathbb{R}, \Delta, \Gamma, (\delta)_{\delta\in\Delta}\big)$ if and only if*

1. *$\delta \in \Gamma \Leftrightarrow \delta' \in G$ for every $\delta \in \Delta$;*
2. *for each $\delta \in \Delta$ and each p, if δ is not a p^{th} power in Δ, then δ' is not a p^{th} power in A ;*
3. *$[p]A = [p]\Delta$ for each p ;*
4. *for all $a_1, \dots, a_n \in \mathbb{Z}$ and $\delta_1, \dots, \delta_n \in \Delta$,*

$$a_1\delta_1 + \cdots + a_n\delta_n > 0 \Longleftrightarrow a_1\delta_1' + \cdots + a_n\delta_n' > 0 \;;$$

5. *$\big(K, A, (\delta')_{\delta\in\Delta}\big)$ satisfies the Mann axioms of Δ.*
6. *$\big(K, G, (\gamma')_{\gamma\in\Gamma}\big)$ satisfies the Mann axioms of Γ.*

Chapter 7 is joint work [10] with Lou van den Dries, except that Section 7.8 is not included in that paper since it was done later. In this chapter we study a uniform version of the Mann property.

We let Ω be an ambient algebraically closed field with a subfield $\mathbf{k}$ and a subgroup Γ of $\Omega^\times$. Call $(\mathbf{k}, \Gamma)$ a *Mann pair* if for each $n \ge 1$ there is a finite $E \subseteq \Gamma^n$ such that for all $a_1, \dots, a_n \in \mathbf{k}^\times$ all non-degenerate solutions of $(*)$ in Γ lie in E.

Suppose $(\mathbf{k}, \Gamma)$ is a Mann pair. It is obvious that then Γ has the Mann property and that if $\mathbf{k}'$ is a subfield of $\mathbf{k}$ and Γ' a subgroup of Γ, then $(\mathbf{k}', \Gamma')$ is also a Mann pair. Taking $n = 1$ in the definition we see that $\mathbf{k}^\times \cap \Gamma$ is finite, and so all elements of $\mathbf{k}^\times \cap \Gamma$ are roots of unity.

The following is a substantial source of Mann pairs, and is proved as Theorem 7.5.3 on page 83:

Theorem 1.0.3. *Suppose Ω has characteristic zero, $\mathbf{k}$ is algebraically closed, $\mathbf{k}^\times \cap \Gamma = \{1\}$, and Γ has finite rank. Then $(\mathbf{k}, \Gamma)$ is a Mann pair.*

This is an analogue for Mann pairs of the result from [11], [16], [21] that was mentioned earlier. In Section 7.2 we first deal with the special case that Γ is finitely generated and $\mathbf{k}(\Gamma)$ has transcendence degree 1 over $\mathbf{k}$; the key here is a height bound by Brownawell and Masser from [6]. With this special case as a stepping stone and using various facts established in Sections 7.3 and 7.4 we prove Theorem 1.0.3 in Section 7.5. This proof has an effective nature, but we do not elaborate on this aspect in this dissertation. Section 7.4 also describes some curious Mann pairs in positive characteristic as well as some Mann pairs $(\mathbb{Q}_p, \Gamma)$ obtained by Teichmüller lifting.

Next we consider the structure $(\Omega, \mathbf{k}, \Gamma)$, that is, the algebraically closed field Ω with distinguished subsets $\mathbf{k}$ and Γ. In Sections 7.3 and 7.6 we show:

Theorem 1.0.4. *The following are equivalent:*

(1) $(\mathbf{k}, \Gamma)$ *is a Mann pair;*

(2) *for each* $n \geq 1$ *the set* $\{(x, y) \in \mathbf{k}^n \times \Gamma^n : x_1y_1 + \cdots + x_ny_n = 0\}$ *is a finite union of sets* $X \times Y$ *where* $X \subseteq \mathbf{k}^n$ *is definable in the field* $\mathbf{k}$ *and* $Y \subseteq \Gamma^n$ *is definable in the group* Γ*;*

(3) *for all* m, n, *every subset of* $\mathbf{k}^m \times \Gamma^n$ *definable in* $(\Omega, \mathbf{k}, \Gamma)$ *is a finite union of sets* $X \times Y$ *with* $X \subseteq \mathbf{k}^m$ *definable in the field* $\mathbf{k}$ *and* $Y \subseteq \Gamma^n$ *definable in the group* Γ*.*

In Section 7.7 we prove that for Mann pairs $(\mathbf{k}, \Gamma)$ with $[\Omega : \mathbf{k}] > 2$, the complete theory $\mathrm{Th}(\Omega, \mathbf{k}, \Gamma)$ is determined by $\mathrm{Th}(\mathbf{k})$ and $\mathrm{Th}(\Gamma)$ after adding names for enough elements of $\mathbf{k}$ and Γ to witness that $(\mathbf{k}, \Gamma)$ is a Mann pair. Under the additional assumption that $\mathbf{k}$ is algebraically closed we show that $(\Omega, \mathbf{k}, \Gamma)$ is stable, and we make further observations along these lines.

The last of our goals in this thesis is the study of the real field with a subgroup Γ of the circle group

$$\mathbb{S} := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}.$$

This situation is studied by Oleg Belegradek and Boris Zilber in [2], but here we consider slightly more general structures, and our techniques differ from theirs.

Instead of a natural *ordering*, subgroups of $\mathbb{S}$ have a compatible *orientation*. We study oriented abelian groups in a more general axiomatic setting, and treat them as we did the dense subgroups of $\mathbb{R}^{>0}$ in [9]: we introduce a notion of *regular density*, in analogy with the regular density for ordered abelian groups. Then we classify regularly dense oriented abelian groups up to elementary equivalence using certain algebraic invariants. Next we identify $\mathbb{C}$ with $\mathbb{R}^2$ in the usual way, via

$$a + bi \mapsto (a, b) : \mathbb{C} \to \mathbb{R}^2 \qquad (a, b \in \mathbb{R}).$$

For an element $\alpha = (\alpha_1, \alpha_2)$ of $\mathbb{C}$, we put $\mathrm{Re}(\alpha) := \alpha_1$ and $\mathrm{Im}(\alpha) := \alpha_2$.
Fix an infinite subgroup Γ of the multiplicative group $\mathbb{S} \subseteq \mathbb{C}^\times$ such that Γ has the Mann property. This makes Γ a subset of $\mathbb{R}^2$. Let $\mathcal{L}_{\mathrm{o}}(P, \Gamma)$ be the language of ordered rings augmented by a binary relation symbol P and by a name for each $\gamma \in \Gamma$.

For every linear equation

$$a_1x_1 + \cdots + a_nx_n = 1 \quad (n \geq 2,\ a_1, \ldots, a_n \in \mathbb{Q}^\times)$$

choose a finite list of its nondegenerate solutions in Γ,

$$\gamma_1 = (\gamma_{11}, \ldots, \gamma_{1n}), \ldots, \gamma_k = (\gamma_{k1}, \ldots, \gamma_{kn}),$$

and let the corresponding *Mann axiom of* Γ be the sentence

$$\forall y \forall z \Big[\big(P(y,z) \wedge \sum_{i=1}^{n} a_i y_i = 1 \wedge \sum_{i=1}^{n} a_i z_i = 0 \wedge \bigwedge_I \big(\sum_{i \in I} a_i y_i \neq 0 \vee \sum_{i \in I} a_i z_i \neq 0 \big) \big) \longrightarrow \bigvee_{j=1}^{k} (y,z) = \gamma_j \Big]$$

in the language $\mathcal{L}_{\mathrm{o}}(P, \Gamma)$, where $y_1, \ldots, y_n, z_1, \ldots, z_n$ are distinct variables, $y = (y_1, \ldots, y_n)$, $z = (z_1, \ldots, z_n)$, $P(y,z)$ abbreviates $P(y_1, z_1) \wedge \cdots \wedge P(y_n, z_n)$, the conjunction $\bigwedge_I$ is over all nonempty proper $I \subsetneq \{1, \ldots, n\}$, "$\sum_{i=1}^{n} a_i y_i = 1$","$\sum_{i=1}^{n} a_i z_i = 0$" "$\sum_{i \in I} a_i y_i \neq 0$", and "$\sum_{i \in I} a_i z_i \neq 0$" represent certain obvious formulas in the language of rings, and $(y, z) = \gamma_j$ abbreviates

$$y_1 = \mathrm{Re}(\gamma_{j1}) \wedge \cdots \wedge y_n = \mathrm{Re}(\gamma_{jn}) \wedge z_1 = \mathrm{Im}(\gamma_{j1}) \wedge \cdots \wedge z_n = \mathrm{Im}(\gamma_{jn}).$$

In Section 8.2 we prove the following (see page 110).

Theorem 1.0.5. *Let K be a real closed field, and let G be an infinite subgroup of*

$$\mathbb{S}(K) := \{(x, y) \in K^2 : x^2 + y^2 = 1\}$$

with a group homomorphism $\Gamma \to G : \gamma \mapsto \gamma'$. Then $(K, G, (\gamma')_{\gamma \in \Gamma}) \equiv (\mathbb{R}, \Gamma, (\gamma)_{\gamma \in \Gamma})$ if and only if

(1) *for every* $\gamma \in \Gamma$ *and* p*, if* γ *is not a* p^{th} *power in* Γ*, then* γ' *is not a* p^{th} *power in* G*,*

(2) $[p]G = [p]\Gamma$ *for all* p*,*

(3) *for all* $\gamma_1, \ldots, \gamma_n \in \Gamma$ *and all polynomials* $Q(X_1, \ldots, X_n) \in \mathbb{Z}[X_1, \ldots, X_n]$:

$$Q\big(\operatorname{Re}(\gamma_1), \ldots, \operatorname{Re}(\gamma_n)\big) > 0 \Leftrightarrow Q\big(\operatorname{Re}(\gamma_1'), \ldots, \operatorname{Re}(\gamma_n')\big) > 0,$$

(4) $\big(K, G, (\gamma')_{\gamma \in \Gamma}\big)$ *satisfies the Mann axioms of* Γ*.*

Chapter 2

Notation, terminology and preliminaries

In this chapter we introduce some notation and terminology that are used later in this thesis, and review some preliminaries from algebra and model theory. We assume familiarity with first order theories, the compactness theorem, saturation, back-and-forth systems, quantifier elimination, types, and model-theoretic algebraic and definable closure. Good references for these are [12] and [20].

Throughout, $\mathbb{N} = \{0, 1, 2, \dots\}$ is the set of natural numbers and m, n (possibly with subscripts or accents) always denote natural numbers. We put $\mathbb{N}_\infty := \mathbb{N} \cup \{\infty\}$.

2.1 Algebra

Let A be an additively written abelian group, and d a positive natural number. We put

$$A[d] := \{a \in A : da = 0\} \quad \text{(the } d\text{-torsion subgroup of } A\text{)},$$

$$A_{\text{tor}} = \{a \in A : na = 0 \text{ for some } n > 0\} \quad \text{(the torsion subgroup of } A\text{)},$$

and

$$[d]A := \begin{cases} |A/dA| & \text{if } A/dA \text{ is finite,} \\ \infty & \text{otherwise,} \end{cases}$$

where $dA := \{da : a \in A\}$.

A subgroup B of A is *pure* if for every $n > 0$, we have $nA \cap B = nB$. A subset X of A is said to be *linearly dependent* if there are distinct $a_1, \dots, a_m \in X$ with $m \geq 1$ and $k_1, \dots, k_m \in \mathbb{Z}$, not all zero, such that $k_1a_1 + \cdots + k_ma_m = 0$; otherwise X is said to be *linearly independent*. The *rank* of A is the cardinality of a maximal linearly independent subset of A. It is well-known that the rank of A equals the dimension of the $\mathbb{Q}$-vector space $A \otimes \mathbb{Q}$, where we consider A as a $\mathbb{Z}$-module and the tensor product is that of $\mathbb{Z}$-modules.

Let A be torsion-free. We put

$$\mathbb{Q}A = \{\frac{a}{m} : \ a \in A, \ m > 0\} \supseteq A,$$

the divisible hull of A. For a subgroup A' of A and a subset X of A, we let $A'\langle X\rangle_A$ be the smallest pure subgroup of A containing both A' and X; its elements are the fractions

$$\frac{a'+k_1x_1+\cdots+k_nx_n}{m}\in\mathbb{Q}A$$

with $a'\in A',\ x_1,\dots,x_n\in X,\ k_1,\dots,k_n\in\mathbb{Z},\ m>0$, such that

$$a'+k_1x_1+\cdots+k_nx_n\in mA.$$

For $x=(x_1,\dots,x_n)\in A^n$ we put $A'\langle x\rangle_A := A'\langle\{x_1,\dots,x_n\}\rangle_A$. When the ambient group A is clear from the context, we write $A'\langle X\rangle$ instead of $A'\langle X\rangle_A$.

Let E be a field. We denote the algebraic closure of E by E^{ac}. We call E *perfect* if either it is of characteristic 0, or for some p it is of characteristic p and every element of E is of the form α^p with $\alpha\in E$. The *perfect closure* of E is the smallest subfield of E^{ac} containing E that is perfect. Hence if E is of characteristic 0, then its perfect closure is itself. If E is of characteristic p, then we let E^{1/p^∞} denote the perfect closure of E.

Other notions we use from field theory are *linear disjointness, freeness*, and *regular extensions*. The definitions of these notions and the proof of the next theorem can be found in [15].

Theorem 2.1.1. *Let* $\mathbf{k}$*,* K*, and* L *be subfields of* E *such that* $\mathbf{k}\subseteq K$*,* $\mathbf{k}\subseteq L$*,* K *is a regular extension of* $\mathbf{k}$*, and* K *and* L *are free over* $\mathbf{k}$*. Then* K *and* L *are linearly disjoint over* $\mathbf{k}$*.*

2.2 Model theory

Throughout this section $\mathcal{L}$ is a many-sorted first order language, with S as its set of sorts. Actually we exploit this generality only in Subsection 5.1.1 and Sections 7.6 and 7.7, and anywhere else the languages in use will be one-sorted. Let $\mathcal{M}=\big((M_s)_{s\in S};\dots\big)$ and $\mathcal{N}=\big((N_s)_{s\in S};\dots\big)$ denote $\mathcal{L}$-structures. We define the *size* of $\mathcal{M}$ to be the cardinal sum

$$|\mathcal{M}| := \sum_{s\in S}|M_s|.$$

We write $\mathcal{M}\subseteq\mathcal{N}$ when $\mathcal{M}$ is a substructure of $\mathcal{N}$. "Definable" means "definable with parameters in the ambient structure" unless specified otherwise.

Let $x=(x_1,\dots,x_n)$ be a tuple of distinct variables, and $\phi(x)$ an $\mathcal{L}$-formula. We denote the set defined by $\phi(x)$ in $\mathcal{M}$ by $\phi(\mathcal{M})$.

Given sets X and Y we write

$$f : X \xrightarrow{n} Y$$

to indicate that f is a map from X to the power set of Y such that $|f(x)| \leq n$ for all $x \in X$. For such an f its *graph* is by definition the set

$$\{(x, y) \in X \times Y : y \in f(x)\},$$

and for $P \subseteq X$ we put $f(P) := \bigcup_{x \in P} f(x)$. If X and Y are definable in the structure $\mathcal{M}$, then such an f is said to be definable in $\mathcal{M}$ if its graph is.

We use the following quantifier elimination criterion (a proof in the one-sorted setting can be found in [12]).

Theorem 2.2.1. *Let T be an $\mathcal{L}$-theory. The following conditions are equivalent:*

(1) *T admits quantifier elimination;*

(2) *for all models $\mathcal{M}, \mathcal{N}$ of T, such that $|\mathcal{M}| \leq |\mathcal{L}|$ and $\mathcal{N}$ is $|\mathcal{L}|^+$-saturated, every embedding of a proper substructure of $\mathcal{M}$ in to $\mathcal{N}$ can be extended to an embedding of some strictly larger substructure of $\mathcal{M}$ into $\mathcal{N}$.*

In this paragraph $\mathcal{L}$ is a one-sorted language. Let U be a unary relation symbol not in $\mathcal{L}$, and let $\mathcal{L}(U)$ be the language $\mathcal{L}$ augmented by U. The *U-restriction* of ϕ is the $\mathcal{L}(U)$-formula ϕ_U defined recursively as follows:

$$\begin{aligned}
&\text{if } \phi \text{ is atomic, then } \phi_U := \phi,\\
&\text{if } \phi = \neg\phi', \text{ then } \phi_U := \neg\phi'_U,\\
&\text{if } \phi = \phi' \wedge \phi'', \text{ then } \phi_U := \phi'_U \wedge \phi''_U,\\
&\text{if } \phi = \phi' \vee \phi'', \text{ then } \phi_U := \phi'_U \vee \phi''_U,\\
&\text{if } \phi = \exists x \phi',\ \text{ then } \phi_U := \exists x (U(x) \wedge \phi'_U),\\
&\text{if } \phi = \forall x \phi',\ \text{ then } \phi_U := \forall x (U(x) \rightarrow \phi'_U).
\end{aligned}$$

Chapter 3

Revisiting the field of real numbers with a small multiplicative group

In the first section of this chapter we recall notation, definitions and results from [9]. The next two sections supplement [9] and use tools from that paper. In section 3.2 we prove some facts about the algebraic and definable closures and definable functions in the algebraically closed case, and in section 3.3 we state a quantifier elimination result in the real closed case which is taken from [3]. Finally, in section 3.4, we give two examples showing that the main theorems in [9] for structures $(\mathbb{R}, G)$ do not generalize to $(\mathbb{R}_{\text{an}}, G)$.

3.1 Notation, definitions, and some facts

It is shown in [9] that if K is an algebraically closed field and G is a subgroup of $K^\times$ with the Mann property, then G is *small* in K in a technical sense: Let $\mathcal{L}$ be a one-sorted first order language, $\mathcal{M} = (M, \dots)$ an $\mathcal{L}$-structure, and $G \subseteq M$. If $f(G^m) = M$ for some m, n and some $f : M^m \xrightarrow{n} M$ definable in $\mathcal{M}$, we call G *large in* $\mathcal{M}$, and otherwise we call G *small in* $\mathcal{M}$. We also say that a set $X \subseteq M$ is *G-bound* (in $\mathcal{M}$) if $X \subseteq f(G^m)$ for some m, n and some $f : M^m \xrightarrow{n} M$ definable in $\mathcal{M}$.

Let $\mathcal{L}(U)$ be the language extending $\mathcal{L}$ by a new unary predicate U. For $\mathcal{M}$ and G as above, let $(\mathcal{M}, G)$ denote the $\mathcal{L}(U)$-expansion of $\mathcal{M}$ obtained by interpreting the symbol U as the subset G of M.
The following generalizes a theorem of Keisler from [13], and is Proposition 2.3 in [9]:

Proposition 3.1.1. *Let T be an $\mathcal{L}$-theory with QE whose models are infinite and strongly minimal. Let $\mathcal{M}$ and $\mathcal{N}$ be models of T with substructures $\mathcal{G} = (G, \dots)$ and $\mathcal{H} = (H, \dots)$, such that*

(i) G is small in $\mathcal{M}$ and H is small in $\mathcal{N}$;

(ii) $\mathcal{G} \equiv \mathcal{H}$ and $\mathcal{M} \equiv \mathcal{N}$.

Then $(\mathcal{M}, G) \equiv (\mathcal{N}, H)$.

If $\mathcal{M}$ is an algebraically closed field, then the many-valued functions in the definition of "large" can be replaced by ordinary functions (Lemma 2.4 in [9]):

Lemma 3.1.2. *Let K be an algebraically closed field and suppose $G \subseteq K$ is large in K. Then there is a function $F : K^{\ell} \to K$, definable in K, such that $F(G^{\ell}) = K$.*

Remark. Suppose F is a subfield of an algebraically closed field K. Then F is large in K if and only if $F = K$, or F is a real closed field and $[K : F] = 2$. This follows from a theorem of E. Artin; see Lemma 3.1 in [13].

From now on in this section K denotes a field with prime field $\mathbb{F}$, and G a subgroup of $K^{\times}$. The next two results are Lemmas 5.12 and 5.13 from [9], and will be used several times in this thesis. We let E be a subfield of K in these lemmas.

Lemma 3.1.3. *Let Γ be a subgroup of G such that for all $a_1, \dots, a_n \in E^{\times}$ the equation $a_1x_1 + \cdots + a_nx_n = 1$ has the same nondegenerate solutions in Γ as in G. Then we have for any $g, g_1, \dots, g_n \in G$:*

(1) *if g is algebraic over $E(\Gamma)$ of degree d, then $g^d \in \Gamma$;*

(2) *if $g_1, \dots, g_n$ are algebraically dependent over $E(\Gamma)$, then they are multiplicatively dependent over Γ.*

Lemma 3.1.4. *Let Γ be as in the previous lemma, and also pure in G. Then $G \cap E(\Gamma) = \Gamma$ and $E(G)$ is a regular field extension of $E(\Gamma)$.*

3.1.1 The algebraically closed case

In this subsection we assume that K is algebraically closed, and that G has the Mann property. Then G is small in K by Lemma 6.1 of [9]. The next result is Corollary 5.2 from [9].

Proposition 3.1.5. *Every subset of G^n definable in (K, G) is definable in the abelian group G.*

Back-and-forth

Now we present a back-and-forth argument that will play a central role in the rest of this section and the next section. We work, however, in a less general situation than [9]; there G is a small multiplicative subset of K rather than a subgroup of $K^{\times}$ with the Mann property.

Suppose that (K, G) is κ-saturated where κ is an uncountable cardinal. Let $\mathrm{Sub}(K, G)$ be the collection of structures (K', G') where K' is an algebraically closed subfield of K of cardinality $< \kappa$, G' is a multiplicative subgroup of $(K')^{\times}$ contained in G such that $G' \preceq G$, and K' and $\mathbb{F}(G)$ are free over $\mathbb{F}(G')$.

Consider an element $(K', G') \in \mathrm{Sub}(K, G)$. Using Lemma 3.1.4 with $\mathbb{F}$ and G' in the place of E and Γ, we get that the field $\mathbb{F}(G)$ is a regular extension of $\mathbb{F}(G')$, and thus K' and $\mathbb{F}(G)$ are linearly disjoint over $\mathbb{F}(G')$. It also follows from the same lemma that $(K', G') \subseteq (K, G)$.

Let L be a second algebraically closed field and H a subgroup of $L^{\times}$ with the Mann property such that (L, H) is κ-saturated, and define $\mathrm{Sub}(L, H)$ likewise. Let $\mathcal{I}$ be the set of all isomorphisms

$$\iota : (K', G') \cong (L', H'),$$

where $(K', G') \in \mathrm{Sub}(K, G)$ and $(L', H') \in \mathrm{Sub}(L, H)$. (Possibly $\mathcal{I} = \emptyset$.) The next lemma and its two corollaries are Lemma 3.4 and Corollaries 3.5 and 3.6 of [9] in our less general setting.

Lemma 3.1.6. *$\mathcal{I}$ is a back-and-forth system.*

Proof. Let $\iota : (K', G') \to (L', H')$ be in $\mathcal{I}$ and let $\alpha \in K \setminus K'$; our task is to find an extension of ι in $\mathcal{I}$ that contains α in its domain. We can assume K and L have the same prime field $\mathbb{F}$. We distinguish three cases:

Case 1: $\alpha \in G$. Take a subgroup G'' of G of cardinality $< \kappa$ such that $G' \cup \{\alpha\} \subseteq G''$, and $G'' \preceq G$. Then $K'(G'')$ and $\mathbb{F}(G)$ are linearly disjoint over $\mathbb{F}(G'')$. Put $K'' := K'(G'')^{\mathrm{ac}}$, so $(K'', G'') \in \mathrm{Sub}(K, G)$.

Since H is κ-saturated, we can extend $\iota|G'$ to an isomorphism $G'' \cong H''$ where H'' is a subgroup of of H such that $H' \subseteq H''$ and $H'' \preceq H$. This isomorphism extends to a field isomorphism

$$\mathbb{F}(G'') \cong \mathbb{F}(H''),$$

and by the linear disjointness of K'' and $\mathbb{F}(G)$ over $\mathbb{F}(G'')$, it extends further to a field isomorphism

$$K'' \cong L'' := L'(H'')^{\mathrm{ac}}$$

that belongs to $\mathcal{I}$.

Case 2: $\alpha \in K'(G)^{\mathrm{ac}}$. Then $\alpha \in K'(g_1, \ldots, g_n)^{\mathrm{ac}}$ for suitable $g_1, \ldots, g_n \in G$. Applying the previous construction n times in succession yields the desired isomorphism in $\mathcal{I}$.

Case 3: $\alpha \notin K'(G)^{\mathrm{ac}}$. Then $(K'(\alpha)^{\mathrm{ac}}, G') \in \mathrm{Sub}(K, G)$. By smallness of H in L, we have $L'(H)^{\mathrm{ac}} \neq L$, so we can choose $\beta \in L \setminus L'(H)^{\mathrm{ac}}$. Then $(L'(\beta)^{\mathrm{ac}}, H') \in \mathrm{Sub}(L, H)$ and ι extends to a field isomorphism $K'(\alpha)^{\mathrm{ac}} \cong L'(\beta)^{\mathrm{ac}}$ that sends α to β and belongs to $\mathcal{I}$. $\square$

Remarks. This proof is used in several places in the rest of this chapter, and it exhibits a template for the back-and-forth arguments that will arise later in other settings. Moreover it gives a way to count the types in (K, G). Let (K', G') be an elementary substructure with $|K'| < \kappa$. Then all elements of K not in $K'(G)^{\mathrm{ac}}$ realize the same type in (K, G) over K'; and if $\alpha \in K'(g_1, \ldots, g_n)^{\mathrm{ac}}$, then the type of α in (K, G)

over K' is completely determined by the type of $(g_1, ..., g_n)$ in the group G over G' and the specification of a polynomial

$$P(X_1, ..., X_n, Y) \in K'[X_1, ..., X_n, Y]$$

such that $P(g_1, ..., g_n, Y) \in K'(g_1, ..., g_n)[Y]$ is irreducible and $P(g_1, ..., g_n, \alpha) = 0$.

Here are some consequences of Lemma 3.1.6 and the remark above:

Corollary 3.1.7. *The structure (K, G) is stable. Moreover (K, G) is ω-stable if G is, and superstable if G is.*

Corollary 3.1.8. *Let K' be an algebraically closed subfield of K with a subgroup G' of $(K')^{\times}$. Suppose $G' \subseteq G$ with $G' \preceq G$, and K' and $\mathbb{F}(G)$ are free over $\mathbb{F}(G')$ in K. Then $(K', G') \preceq (K, G)$.*

We have some more information in the case that G is ω-stable (Theorem 6.4 of [9]):

Theorem 3.1.9. *Suppose G is infinite and ω-stable. Then $\mathrm{MR}(G^n) = n$ and $\mathrm{MR}(K) = \omega$, where the Morley rank is with respect to the ω-stable theory $\mathrm{Th}(K, G)$.*

Quantifier elimination

We get a relative quantifier elimination after assuming that $G/G^{[d]}$ is finite for every positive integer d. The next result is Proposition 6.7 in [9]; we let $x_1, \dots, x_m, y_1, \dots, y_n$ be distinct variables and we put $x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_n)$.

Proposition 3.1.10. *Suppose $G/G^{[d]}$ is finite for each integer $d > 0$. Then every subset of K^m definable in (K, G) is a boolean combination of subsets of K^m defined by formulas $\exists y\big(U(y) \wedge \phi(x, y)\big)$ where $\phi(x, y)$ is a quantifier-free formula in the language of rings augmented by names for elements of K.*

Remark. A closer look at the proof of 3.1.10 in [9] gives some more information on the parameters in $\phi(x, y)$: Let $\psi(x)$ be a formula in the language of rings augmented by a unary predicate and names for elements of a subset A of K, defining $X \subseteq K^m$. Then $\phi(x, y)$ as in 3.1.10 can be chosen to be a quantifier-free formula in the language of rings augmented by names for the elements of $A \cup G$.

3.1.2 The real case

Let Γ be a subgroup of $\mathbb{R}^{>0}$ with the Mann property. Examples to keep in mind are $\Gamma = \{1\}$, $\Gamma = 2^{\mathbb{Z}}$, and $\Gamma = 2^{\mathbb{Z}}3^{\mathbb{Z}}$. Let $\mathcal{L}_{\mathrm{o}}(U, \Gamma)$ be the language of ordered rings augmented by a unary relation symbol U and by a name (constant symbol) γ for each element $\gamma \in \Gamma$. The *ordering axioms* of Γ are the following: given any tuple $k = (k_1, \dots, k_n)$ of integers, and any tuple $\gamma = (\gamma_1, \dots, \gamma_n)$ of elements of Γ, $n \geq 1$, the ordering axiom for k, γ is the sentence

$$k_1\gamma_1 + \cdots + k_n\gamma_n > 0$$

if this sentence holds in $\mathbb{R}$, and otherwise it is the sentence

$$k_1\gamma_1 + \cdots + k_n\gamma_n \leq 0.$$

Let $\mathrm{RCF}(\Gamma)$ be the theory in the language $\mathcal{L}_{\mathrm{o}}(U, \Gamma)$ whose models are the structures $\big(K, G, (\gamma')_{\gamma \in \Gamma}\big)$ such that

(1) K is a real closed ordered field, and G is a dense subgroup of $K^{>0}$,

(2) $\gamma \mapsto \gamma' \ : \Gamma \to G$ is a group homomorphism,

(3) $\big(K, (\gamma')_{\gamma \in \Gamma}\big)$ satisfies the ordering axioms of Γ,

(4) $\big(K, G, (\gamma')_{\gamma \in \Gamma}\big)$ satisfies the Mann axioms of Γ.

Here γ' is the interpretation of (the name of) γ in $\big(K, G, (\gamma')_{\gamma \in \Gamma}\big)$. If Γ is dense in $\mathbb{R}^{>0}$, then $\big(\mathbb{R}, \Gamma, (\gamma)_{\gamma \in \Gamma}\big)$ is clearly a model of $\mathrm{RCF}(\Gamma)$. The theory $\mathrm{RCF}(\Gamma)$ is not complete; we shall classify its models up to elementary equivalence. We have for any model $\big(K, G, (\gamma')_{\gamma \in \Gamma}\big)$ of $\mathrm{RCF}(\Gamma)$ a unique ordered field embedding $\mathbb{Q}(\Gamma) \to K$ that sends each $\gamma \in \Gamma$ to γ', and from now on we identify $\mathbb{Q}(\Gamma)$ with an ordered subfield of K via this embedding, and accordingly write γ instead of γ'. Next is the the main result of this section, which is Theorem 7.1 in [9]. We use its proof in Section 3.3.

Theorem 3.1.11. *Let* $\big(K, G, (\gamma)\big)$ *and* $\big(L, H, (\gamma)\big)$ *be models of* $\mathrm{RCF}(\Gamma)$. *Then* $\big(K, G, (\gamma)\big) \equiv \big(L, H, (\gamma)\big)$ *if and only if* $[p]G = [p]H$ *for every* p, *and for each* $\gamma \in \Gamma$ *and each* $n > 0$:

$$\gamma \textit{ is an } n^{\text{th}} \textit{ power in } G \iff \gamma \textit{ is an } n^{\text{th}} \textit{ power in } H.$$

Proof. The "only if" part is obvious. For the "if" part, assume that $[p]G = [p]H$ for every p, and that for each $\gamma \in \Gamma$ and $n > 0$:

$$\gamma \text{ is an } n^{\text{th}} \text{ power in } G \iff \gamma \text{ is an } n^{\text{th}} \text{ power in } H.$$

We can assume also that $\big(K, G, (\gamma)\big)$ and $\big(L, H, (\gamma)\big)$ are κ-saturated where κ is an uncountable cardinal larger than $|\Gamma|$. Let $\mathrm{Sub}(K, G)$ be the collection of $\mathcal{L}_{\mathrm{o}}(U)$-structures (K', G') where K' is a real closed ordered subfield of K of cardinality $< \kappa$, $G' \subseteq K'^{>0}$ is a pure subgroup of G containing Γ, and K' and $\mathbb{Q}(G)$

are free over $\mathbb{Q}(G')$. Note that if $(K', G') \in \mathrm{Sub}(K, G)$, then by Lemma 3.1.4 we have $(K', G') \subseteq (K, G)$, the field $\mathbb{Q}(G)$ is a regular extension of $\mathbb{Q}(G')$, and thus by Theorem 2.1.1, K' and $\mathbb{Q}(G)$ are linearly disjoint over $\mathbb{Q}(G')$.

We define $\mathrm{Sub}(L, H)$ likewise. Let $\mathcal{I}$ be the set of all isomorphisms $\iota : (K', G') \cong (L', H')$ where $(K', G') \in \mathrm{Sub}(K, G)$, $(L', H') \in \mathrm{Sub}(L, H)$, and $\iota(\gamma) = \gamma$ for all $\gamma \in \Gamma$. We first show that $\mathcal{I} \neq \emptyset$: Let

$$G' := \{g \in G : g^n \in \Gamma \text{ for some } n > 0\}, \quad K' := \mathbb{Q}(\Gamma)^{\mathrm{rc}} \subseteq K$$
$$H' := \{h \in H : h^n \in \Gamma \text{ for some } n > 0\}, \quad L' := \mathbb{Q}(\Gamma)^{\mathrm{rc}} \subseteq L.$$

Then $(K', G') \in \mathrm{Sub}(K, G)$ and $(L', H') \in \mathrm{Sub}(L, H)$, and the ordered field isomorphism $K' \cong L'$ that is the identity on Γ belongs to $\mathcal{I}$.

To establish the theorem, it remains to prove:

Claim. $\mathcal{I}$ is a back-and-forth system.

To prove the claim, let $\iota : (K', G') \to (L', H')$ be in $\mathcal{I}$ and let $\alpha \in K \setminus K'$; our task is to find an extension of ι in $\mathcal{I}$ that contains α in its domain. We distinguish three cases:

Case 1: $\alpha \in G$. Using the main theorem of [22], we can pick a $\beta \in H$ such that for all $x \in G'$, $k \in \mathbb{Z}$ and $n > 0$ we have:

$$\alpha^k x \text{ is an } n^{\text{th}} \text{ power in } G \iff \beta^k \iota(x) \text{ is an } n^{\text{th}} \text{ power in } H.$$

We shall now adjust this β to make it realize the cut in L' that corresponds under ι to the cut of α in K'. By saturation, this cut in L' is realized in L by all elements of an entire interval (s, t) where $0 < s < t$ in L. Since H is dense in $L^{>0}$ we can assume that $s, t \in H$. Also by saturation, the elements $h \in H$ that are n^{th} powers in H for each $n > 0$ lie dense in H. Hence for a suitable such h we have $s < h\beta < t$. Put

$$K'' := K'(\alpha)^{\mathrm{rc}}, \qquad G'' = K'' \cap G,$$
$$L'' := L'(h\beta)^{\mathrm{rc}}, \qquad H'' = L'' \cap H.$$

By the extension procedure for regularly dense groups from [22], and by Lemmas 3.1.3 and 3.1.4 we have $(K'', G'') \in \mathrm{Sub}(K, G)$, $(L'', H'') \in \mathrm{Sub}(L, H)$ and an isomorphism $(K'', G'') \to (L'', H'')$ that takes α to $h\beta$.

Case 2: $\alpha \in K'(G)^{\mathrm{rc}}$. Then $\alpha \in K'(g_1, \ldots, g_n)^{\mathrm{rc}}$ for some $g_1, \ldots, g_n \in G$, so we can extend ι by applying Case 1 n times.

Case 3: $\alpha \notin K'(G)^{\mathrm{rc}}$. As in the first case, we consider the cut of α in K'. By saturation and smallness of H we have $L'(H)^{\mathrm{rc}} \neq L$, so there is a $\beta \in L \setminus L'(H)^{\mathrm{rc}}$ that realizes the corresponding cut in L'. By linear disjointness, we can extend ι to an isomorphism $(K'', G') \to (L'', H')$ sending α to β, where $K'' = K'(\alpha)^{\mathrm{rc}}$ and $L'' = L'(\beta)^{\mathrm{rc}}$. In this case the freeness of K'' and $\mathbb{Q}(G)^{\mathrm{rc}}$ over $\mathbb{Q}(G')^{\mathrm{rc}}$ follows from the assumption that $\alpha \notin K'(G)^{\mathrm{rc}}$. $\square$

For divisible G, Theorem 7.2 of [9] describes the structure induced on G by (K, G) as follows:

Theorem 3.1.12. *Suppose G is divisible and $X \subseteq G^m$. Then X is definable in (K, G) if and only if $X = Y \cap G^m$ for some $Y \subseteq K^m$ definable in K.*

We refine this theorem in Section 3.3. The following consequence of Theorem 3.1.12 is Corollary 7.3 in [9].

Corollary 3.1.13. *If G is divisible, then G with its induced structure is weakly o-minimal, even in the strong sense that its theory is weakly o-minimal. (By "G with its induced structure" we mean the ordered group G equipped with the relations $X \subseteq G^m$ that are definable in (K, G), $m = 1, 2, \ldots$.)*

We also note the following converse of the theorem above:

Corollary 3.1.14. *Suppose every set $X \subseteq G$ definable in (K, G) is of the form $Y \cap G$ with $Y \subseteq K$ definable in K. Then G is divisible.*

This is because the assumption here implies weak o-minimality of the ordered group G, which in turn implies the divisibility of G.

Next we state the real closed field analogue of Theorem 3.1.10 (Theorem 7.5 in [9]):

Theorem 3.1.15. *Suppose that $[p]G$ is finite for each p. Then every subset of K^m definable in (K, G) is a boolean combination of subsets of K^m defined in (K, G) by formulas $\exists y\big(U(y) \wedge \phi(x, y)\big)$ where $\phi(x, y)$ is a quantifier-free formula in the language of ordered rings augmented by names for elements of K.*

3.2 Algebraic closure, definable closure and definable functions

This section is inspired by Section 3 in [8]. Here we return to the assumptions of Subsection 3.1.1, so K is an algebraically closed field with prime field $\mathbb{F}$ and G is a subgroup of $K^\times$ with the Mann property. Suppose also that (K, G) is κ-saturated, where κ is an uncountable cardinal. We let $A \subseteq K$ denote a parameter set, and $\operatorname{acl}(A)$ denotes the model-theoretic algebraic closure of A in (K, G). Likewise $\operatorname{dcl}(A)$ denotes the model-theoretic definable closure of A in (K, G).

We start with a very general remark.

Remark. Let $X \subseteq Y \subseteq K$. Construe (K, X) as a structure in the language of rings augmented by a unary predicate U, by interpreting this new predicate as X. Then a field automorphism of K over Y is an automorphism of the structure (K, X) over Y.

Now taking X to be G and Y to be $A \cup G$ in the remark above, we get the following.

Corollary 3.2.1. *The model-theoretic algebraic closure,* $\mathrm{acl}(A \cup G)$*, of* $A \cup G$ *in* (K, G) *is* $\mathbb{F}(A \cup G)^{\mathrm{ac}}$.

Proof. It is clear that $\mathbb{F}(G \cup A)^{\mathrm{ac}} \subseteq \mathrm{acl}(G \cup A)$. For the other inclusion, let $\alpha \notin \mathbb{F}(G \cup A)^{\mathrm{ac}}$. Then the set

$$\{\beta \in K : \beta = f(\alpha) \text{ for some automorphism } f \text{ of } K \text{ over } A \cup G\}$$

is infinite. Thus by the previous remark the set

$$\{\beta \in K : \beta = f(\alpha) \text{ for some automorphism } f \text{ of } (K, G) \text{ over } A \cup G\}$$

is infinite. Hence $\alpha \notin \mathrm{acl}(A \cup G)$.

□

It follows from this corollary that the operator acl_G defined on the subsets of K by $\mathrm{acl}_G(A) := \mathrm{acl}(G \cup A)$ is a pregeometry.

The following direct consequence of Corollary 3.1.8 fits this section and gets used in the lemma after it.

Lemma 3.2.2. *Let* K' *be an algebraically closed subfield of* K*, and* G' *a subgroup of* $(K')^{\times}$ *contained in* G*. Suppose also that* $G' \preceq G$*, and that* K' *and* $\mathbb{F}(G)$ *are free over* $\mathbb{F}(G')$*. Then* K' *is algebraically closed in* (K, G) *in the model-theoretic sense.*

Lemma 3.2.3. *Let* K' *be an algebraically closed subfield of* K*, and* G' *a subgroup of* $(K')^{\times}$ *contained in* G *such that* $G' \preceq G$ *and* K' *and* $\mathbb{F}(G)$ *are free over* $\mathbb{F}(G')$*. Suppose that* $\alpha \in K \setminus K'(G)^{\mathrm{ac}}$*. Then* $K'(\alpha)^{\mathrm{ac}}$ *is algebraically closed in* (K, G)*.*

Proof. It is easy to see that $K'(\alpha)$ and $K'(G)$ are linearly disjoint over K'. It is also true that K' and $\mathbb{F}(G)$ are linearly disjoint over $\mathbb{F}(G')$. Thus it follows that $K'(\alpha)^{\mathrm{ac}}$ and $\mathbb{F}(G)$ are free over $\mathbb{F}(G')$. Then using the previous lemma with $K'(\alpha)^{\mathrm{ac}}$ in the place of K', we get that $K'(\alpha)^{\mathrm{ac}}$ is algebraically closed in the structure (K, G). □

Now we consider the definable closure of certain sets. An argument similar to the proof of Corollary 3.2.1 gives the following.

Corollary 3.2.4. *The definable closure* $\mathrm{dcl}(A \cup G)$*, of* $A \cup G$ *in* (K, G) *is the perfect closure of* $\mathbb{F}(A \cup G)$ *in the field* K*.*

Lemma 3.2.5. *Let* K' *be an algebraically closed subfield of* K*, and* G' *a subgroup of* $(K')^{\times}$ *contained in* G *such that* $G' \preceq G$ *and* K' *and* $\mathbb{F}(G)$ *are free over* $\mathbb{F}(G')$*, and let* $\alpha \in K \setminus K'(G)^{\mathrm{ac}}$*. Then the perfect closure of* $K'(\alpha)$ *in* K *is definably closed in* (K, G)

Proof. Let K'' denote the perfect closure of $K'(\alpha)$. As in the proof of Lemma 3.2.3 it is again the case that $K'(\alpha)^{\mathrm{ac}}$ and $K'(G)$ are linearly disjoint over K'.

Now let $\beta \in K \setminus K''$. If $\beta \notin K'(\alpha)^{\mathrm{ac}}$, then by Lemma 3.2.3, $\beta \notin \mathrm{acl}(K'')$. Thus we let $\beta \in K'(\alpha)^{\mathrm{ac}} \setminus K''$. Then take a field automorphism of $K'(\alpha)^{\mathrm{ac}}$ fixing K'' and moving β. Then by the linear disjointness of $K'(\alpha)^{\mathrm{ac}}$ and $K'(G)$ over K', we get a field automorphism of K fixing $K'(G)$ and moving β. Thus by the first remark of this subsection, we get that $\beta \notin \mathrm{dcl}(K'(\alpha))$. $\square$

Corollary 3.2.6. *Let* $F : K \to K$ *be a function definable in* (K, G)*. Then there is a* G*-bound* $X \subseteq K$ *such that* X *is definable in* (K, G) *and* F *agrees on* $K \setminus X$ *with a function* $K \to K$ *that is definable in* K*.*

Proof. Take a $|K|^{+}$-saturated elementary extension (K^*, G^*) of (K, G), and let $\alpha^* \in K^*$. Then $\alpha^* \in \bigcup_f f\big((G^*)^m\big)$ if and only if $\alpha^* \in K(G^*)^{\mathrm{ac}}$, where f is ranging over n-valued functions from $(K^*)^m$ to K^*(with various m, n) that are definable over K in the field K^* . This means α^* is not in X^* for every G-bound subset X of K if and only if $\alpha^* \notin K(G^*)^{\mathrm{ac}}$, where X^* is the set of realizations in K^* of a formula defining X in (K, G). Thus, in order to prove the corollary, it suffices to show that if $\alpha^* \in K^* \setminus K(G^*)^{\mathrm{ac}}$, then $F(\alpha^*)$ is in the perfect closure of $K(\alpha^*)$. That follows from the previous lemma, taking K^*, G^*, K, G, α^* in the place of K, G, K', G', α. $\square$

Corollary 3.2.7. *Let* $Y \subseteq K$ *be definable in* (K, G)*. Then there is a* G*-bound subset* X *of* K *such that* X *is definable in* (K, G) *and* $Y \setminus X = Y' \setminus X$ *for some* Y' *definable in* K*.*

Proof. Apply the previous corollary to the characteristic function of Y. $\square$

Combining Theorem 3.1.9, and the proof of Corollary 3.2.6, we have the following:

Corollary 3.2.8. *Suppose that* G *is* ω*-stable and let* $X \subseteq K$ *be definable in* (K, G)*. Then* X *is* G*-bound in* K *if and only if it is of finite Morley rank in* (K, G)*.*

Proof. Using Theorem 3.1.9, it is clear that if X is G-bound, then it is of finite Morley rank as it is contained in the image of G^m for some m under a definable n-valued map.

For the other implication assume that X is not G-bound. Take a $|K|^+$-saturated elementary extension (K^*, G^*) of (K, G), and let X^* be the definable subset of $(K^*)^n$ defined by the formula defining X in (K, G). By saturation there is $x^* \in X^* \setminus \bigcup_f f((G^*)^m)$, where f is ranging over n-valued functions from $(K^*)^m$ to K^* that are definable over K in the field K^*. Since $\bigcup_f f((G^*)^m) = K(G^*)^{\mathrm{ac}}$, $x^* \notin K(G^*)^{\mathrm{ac}}$. But then $\mathrm{MR}(x^*) = \omega$. Thus $\mathrm{MR}(X) = \mathrm{MR}(X^*) = \omega$. □

3.3 Yet another QE result

Here we refine Theorem 3.1.12 as promised. We observed Proposition 3.3.1 in [3], where it plays an essential role to show that the field of real numbers expanded by a dense subgroup of $\mathbb{R}^{>0}$ with the Mann property is super rosy of thorn rank ω; an example of such an expansion is $(\mathbb{R}, 2^{\mathbb{Q}})$.

In this subsection we assume that K is a real closed field, and G is a dense subgroup of $K^{>0}$. Remember that the assumption that G has the Mann property is still valid.

For a prime power $q := p^m$ and a tuple $\vec{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ define

$$D_{q,\vec{k}}(G) := \{(g_1, \ldots, g_n) \in G^n : x_1^{k_1} \cdots x_n^{k_n} \in G^{[q]}\} \subseteq G^n.$$

First note that for every q, $D_{q,\mathbf{0}} = G^n$, where $\mathbf{0} := (0, \ldots, 0) \in \mathbb{Z}^n$. For $\vec{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ let

$$\chi_{\vec{k}} : G^n \to G$$

be the group morphism taking $(g_1, \ldots, g_n)$ to $g_1^{k_1} \cdots g_n^{k_n}$. Then $D_{q,\vec{k}}(G)$ is the preimage of the subgroup $G^{[q]}$ of G under $\chi_{\vec{k}}$. Hence $D_{q,\vec{k}}(G)$ is a subgroup of G^n, and there is an induced group embedding

$$G^n / D_{q,\vec{k}}(G) \to G/G^{[p]}.$$

It follows that $|G^n / D_{q,\vec{k}}(G)| \leq |G/G^{[q]}|$ for every $\vec{k} \in \mathbb{Z}^n$.

For $g \in G$ consider the set

$$\{\vec{g} \in G^n : (g, \vec{g}) \in D_{q,\vec{k'}}(G)\},$$

where $\vec{k'} = (1, k_1, \ldots, k_n) \in \mathbb{Z}^{n+1}$. This set is the preimage of the coset $g^{-1}G^{[q]}$ of $G^{[q]}$ in G under $\chi_{\vec{k}}$. Hence it is a coset of $D_{q,\vec{k}}(G)$ in G^n.

If G is divisible, then $D_{q,\vec{k}}(G) = G^n$ for every $q, \vec{k}$. Thus the following proposition is really a generalization of Theorem 3.1.12.

Proposition 3.3.1. *Let $X \subseteq G^n$ be definable in (K, G). Then X is a boolean combination of definable sets of the form $Y \cap \vec{g}D_{q,\vec{k}}(G)$ where Y is definable in the field K, q as above, $\vec{k} \in \mathbb{Z}^n$, and $\vec{g} \in G^n$.*

Proof. By standard model-theoretic arguments (see for instance 8.4.1 of [12]), it is enough to prove the following:

Claim. Let (K_1, G_1) and (K_2, G_2) be two $|K|^+$-saturated elementary extensions of (K, G). Take $\vec{g}_1 \in G_1^n$ and $\vec{g}_2 \in G_2^n$ such that for every formula $\varphi(\vec{x})$ in the language of ordered rings with parameters from K, for every $g \in G$, and for every q, $\vec{k}$ as above, we have

$$K_1 \models \varphi(\vec{g}_1) \text{ and } (g, \vec{g}_1) \in D_{q,\vec{k'}}(G_1) \text{ iff } K_2 \models \varphi(\vec{g}_2) \text{ and } (g, \vec{g}_2) \in D_{q,\vec{k'}}(G_2),$$

where $\vec{k'} = (1, k_1, \dots, k_n) \in \mathbb{Z}^{n+1}$. Then $\vec{g}_1$ and $\vec{g}_2$ realize the same types over K in (K_1, G_1) and (K_2, G_2) respectively.

Proof of the claim. Note that (K, G), (K_1, G_1) and (K_2, G_2) are models of RCF(G), and there is a back and forth system, $\mathcal{I}$, between (K_1, G_1) and (K_2, G_2) as mentioned in the previous section, containing the identity function on (K, G). It suffices to prove that there is an element ι of $\mathcal{I}$ taking $\vec{g}_1$ to $\vec{g}_2$.

Since $\vec{g}_1$ and $\vec{g}_2$ satisfy the same ordered field type over K, there is an ordered field isomorphism

$$\iota : K_1' \to K_2',$$

fixing K and mapping $\vec{g}_1$ to $\vec{g}_2$, where $K_i' := K(\vec{g}_i)^{\text{rc}} \subseteq K_i$ for $i = 1, 2$.

Consider $G_i' := K_i' \cap G_i$. We wish to show that G_i' equals

$$G\langle \vec{g}_i \rangle := \{(g\chi_{\vec{k}}(\vec{g}_i))^{1/m} \in G_i : g \in G, \vec{k} \in \mathbb{Z}^n, m > 0, g\chi_{\vec{k}}(\vec{g}_i) \in G_i^{[m]}\}.$$

It is clear that $G_i' \supseteq G\langle \vec{g}_i \rangle$.

We use Lemma 3.1.3 to show $G_i' \subseteq G\langle \vec{g}_i \rangle$. To do this we need to check that for all $a_1, \dots a_n \in K$, if $a_1 x_1 + \cdots + a_k x_n = 1$ has a nondegenerate solution in G_i, then this solution lies in $G\langle \vec{g}_i \rangle$. But since $(K, G) \preceq (K_i, G_i)$, such a solution lies even in G. Now applying Lemma 3.1.3 with $G\langle \vec{g}_i \rangle$ in the place of Γ, we see that if $g \in G_i$ is algebraic of degree d over $K(G\langle \vec{g}_i \rangle)$, then g^d is in $G\langle \vec{g}_i \rangle$ and thus g itself is in $G\langle \vec{g}_i \rangle$.

Let $g' \in G_1'$ and take $g \in G$, $\vec{k} \in \mathbb{Z}^n$, and $m > 0$ such that $(g\chi_{\vec{k}}(\vec{g}_1))^{1/m}$. Note that

$$\iota((g\chi_{\vec{k}}(\vec{g}_1))^{1/m}) = (g\chi_{\vec{k}}(\vec{g}_2))^{1/m},$$

and by our assumption on $\vec{g}_i$, $g\chi_{\vec{k}}(\vec{g}_1)$ is in $G_1^{[m]}$ if and only if $g\chi_{\vec{k}}(\vec{g}_2)$ is in $G_2^{[m]}$. Hence $\iota(G_1') = G_2'$, and thus ι is an isomorphism between (K_1', G_1') and (K_2', G_2').

It remains to show that K_i' and $\mathbb{Q}(G_i)$ are free over $\mathbb{Q}(G_i')$ and G_i' is a pure subgroup of G_i. The first follows from the assumption that (K_i, G_i) is an elementary extension of (K, G), and G_i' is a pure subgroup of G_i, since it equals $G\langle\vec{g}_i\rangle$.

□

3.4 Examples

In this section we present two examples illustrating that the results of [9] do not go through when the real field $\mathbb{R}$ is replaced by its polynomially bounded o-minimal expansion $(\mathbb{R}, \exp|_{[-1,1]})$. In what follows

$$f(x) := \begin{cases} \log_2(x) & \text{if } 1/2 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

and $\mathbb{R}_{\text{an}}$ denotes the expansion of the real field $\mathbb{R}$ by restricted analytic functions. It is clear that f is definable in $\big(\mathbb{R}, \exp|[-1,1]\big)$, and thus in $\mathbb{R}_{\text{an}}$.

Example 1. Consider the expansion $(\mathbb{R}_{\text{an}}, 2^{\mathbb{Q}})$ of $\mathbb{R}_{\text{an}}$. The image of $2^{\mathbb{Q}}$ under f is $\mathbb{Q} \cap [-1,1]$. Then by taking reciprocals we can define the set $\mathbb{Q} \subseteq \mathbb{R}$ in $(\mathbb{R}_{\text{an}}, 2^{\mathbb{Q}})$. Therefore the set $\mathbb{Z}$ is definable in $(\mathbb{R}_{\text{an}}, 2^{\mathbb{Q}})$ by Theorem 3.1 of [23]. This means there is no hope to get model-theoretic results for that structure analogous to the ones in [9].

Example 2. This time expand $\mathbb{R}_{\text{an}}$ by $2^{\mathbb{Q}}3^{\mathbb{Q}}$. Let $X := (\mathbb{Q} + \mathbb{Q}\alpha) \cap [-1,1]$, where $\alpha := \log_2 3$. Then X is the image of $2^{\mathbb{Q}}3^{\mathbb{Q}}$ under f, so X is definable in $(\mathbb{R}_{\text{an}}, 2^{\mathbb{Q}}3^{\mathbb{Q}})$. Now let $Y := \{x \in X : xX \subseteq X\} \subseteq X$. It is clear that $\mathbb{Q} \cap [-1,1] \subseteq Y$. We claim that $\mathbb{Q} \cap [-1,1] = Y$. To see this, let $x \in Y$, and take $q, r \in \mathbb{Q}$ such that $x = q + r\alpha$. Since $\alpha/2 \in X$, this gives $x\,\alpha/2 = (q\,\alpha + r\,\alpha^2)/2 \in X$. Since α is transcendental over $\mathbb{Q}$, we get $r = 0$, and this proves the claim. Thus the set $\mathbb{Q} \subseteq \mathbb{R}$ is definable in $(\mathbb{R}_{\text{an}}, 2^{\mathbb{Q}}3^{\mathbb{Q}})$ and so is the set $\mathbb{Z} \subseteq \mathbb{R}$ again by Theorem 3.1 of [23].

In contrast to these negative results is the fact that $(\mathbb{R}_{\text{an}}, 2^{\mathbb{Z}})$ behaves like $(\mathbb{R}, 2^{\mathbb{Z}})$ with respect to definable sets; see [19].

Chapter 4

Pairs of abelian groups

In this chapter we study pairs of abelian groups. The first section is devoted to the pure algebraic setting; we prove the material needed in the rest of the present chapter and the next. The second section concentrates on the model theory of pairs of abelian groups. Finally in the last section we prove model completeness for a certain theory of abelian groups, which gets used in Chapter 7.

4.1 Some group theoretic facts

In this section A is an (additively written) abelian group. We let a, sometimes decorated with a subscript or accent, range over A.

Let $\mathcal{A}$ be a collection of subgroups of A. We say that $\mathcal{A}$ is a *lattice* if $A_1 \cap A_2$ and $A_1 + A_2$ are in $\mathcal{A}$ whenever A_1, A_2 are. We say a lattice $\mathcal{A}$ is *distributive* if for all $A_1, A_2, A_3 \in \mathcal{A}$, we have

$$(A_1 \cap A_2) + A_3 = (A_1 + A_3) \cap (A_2 + A_3).$$

If A_1, A_2, A_3 are subgroups of A, then one always has

$$(A_1 \cap A_2) + A_3 \subseteq (A_1 + A_3) \cap (A_2 + A_3).$$

So in order to check that a lattice $\mathcal{A}$ is distributive, it suffices to show that

$$(A_1 \cap A_2) + A_3 \supseteq (A_1 + A_3) \cap (A_2 + A_3)$$

for all $A_1, A_2, A_3 \in \mathcal{A}$. If $\mathcal{A}$ is a distributive lattice, then by induction it follows that for all $A_1, \dots, A_k, A_{k+1}$ from $\mathcal{A}$,

$$\Big(\bigcap_{i=1}^{k} A_i\Big) + A_{k+1} = \bigcap_{i=1}^{k} (A_i + A_{k+1}). \tag{4.1.1}$$

We do not need this, but if $\mathcal{A}$ is a distributive lattice, then

$$(A_1 + A_2) \cap A_3 = (A_1 \cap A_3) + (A_2 \cap A_3)$$

for all $A_1, A_2, A_3 \in \mathcal{A}$; see Theorem 4 in Chapter XI, Section 7 of [4].

The next lemma is a well-known generalization of the Chinese Remainder Theorem.

Lemma 4.1.1. *Let $\mathcal{A}$ be a distributive lattice, with $A_1, \dots, A_k \in \mathcal{A}$ and $a_1, \dots, a_k \in A$. Then the following conditions are equivalent:*

(i) there is a such that $a \equiv a_i \mod A_i$ for $i = 1, \dots, k$;

(ii) $a_i \equiv a_j \mod (A_i + A_j)$ for all $i, j \in \{1, \dots, k\}$.

Proof. If there is $a \in A$ such that $a \equiv a_i \mod A_i$ for $i = 1, \dots, k$, then $a_i \equiv a_j \mod (A_i + A_j)$ for all $i, j \in \{1, \dots, k\}$. For the other direction we proceed by induction on k. There is nothing to prove in the case $k = 1$, so let $k \geq 2$, and suppose that $a_i \equiv a_j \mod (A_i + A_j)$ for all $i, j \in \{1, \dots, k\}$. Our task is to find a congruent to a_i modulo A_i for $i = 1, \dots, k$. The induction hypothesis gives a' such that $a' \equiv a_i \mod A_i$ for $i = 1, \dots, k-1$. It follows that $a' \equiv a_k \mod (A_i + A_k)$ for $i = 1, \dots, k-1$. Hence

$$a' - a_k \in \bigcap_{i=1}^{k-1} (A_i + A_k).$$

Thus by (4.1.1) we have $a' - a_k \in \left(\bigcap_{i=1}^{k-1} A_i\right) + A_k$. Say $a' - a_k = c + d$, where $c \in \bigcap_{i=1}^{k-1} A_i$ and $d \in A_k$. Now let $a = a' - c$. Then $a \equiv a_i \mod A_i$ for $i = 1, \dots, k$, as desired. □

We apply the lemma to the collection $\mathcal{A}$ of subgroups of A of the form mA. We claim that

$$mA + nA = \gcd(m,n)A, \qquad mA \cap nA = \operatorname{lcm}(m,n)A.$$

This is obvious when $m = n = 0$, with $\gcd(0,0) := 0 =: \operatorname{lcm}(0,0)$. Assume $m \neq 0$ or $n \neq 0$ and put $\gamma := \gcd(m,n)$, so $m = m_1\gamma$ and $n = n_1\gamma$, and $xm_1 + yn_1 = 1$ with integers x, y. This easily yields the identity on the left, and the one on the right follows likewise, using $\operatorname{lcm}(m,n) = m_1 n_1 \gamma$. Hence $\mathcal{A}$ is a lattice, and $\mathcal{A}$ is distributive because the lattice of subgroups of $\mathbb{Z}$ is distributive. Thus:

Corollary 4.1.2. *Given $a_1, \dots, a_k$ and $m_1, \dots, m_k$, the following conditions are equivalent:*

(i) there is a such that $a \equiv a_i \mod m_iA$ for $i = 1, \dots, k$;

(ii) $a_i \equiv a_j \mod \gcd(m_i, m_j)A$ for all $i, j \in \{1, \dots, k\}$.

We now fix a subgroup G of A, and let g, sometimes subscripted or otherwise decorated, range over G. Note that a subgroup $mA+G$ of A corresponds to the subgroup $m\bar{A}$ of $\bar{A} := A/G$. Therefore the collection of subgroups of A of the form $mA+G$ is a distributive lattice, and for all $m_1, \dots, m_k$,

$$\bigcap_{i=1}^{k}(m_iA+G) = \operatorname{lcm}(m_1, \dots, m_k)A + G = \big(\bigcap_{i=1}^{k} m_iA\big) + G. \tag{4.1.2}$$

Lemma 4.1.3. *Let $a_1, \dots, a_k$ and $m_1, \dots, m_k$ be given. Then the following conditions are equivalent:*

(i) there exists g such that $g \equiv a_i \mod m_iA$ for $i = 1, \dots, k$;

(ii) for all $i, j \in \{1, \dots, k\}$, $a_i \equiv a_j \mod m_iA + m_jA$ and $a_i \equiv 0 \mod (m_iA+G)$.

Proof. If $g \equiv a_i \mod m_iA$ for $i = 1, \dots, k$, then obviously

$$a_i \equiv a_j \mod m_iA + m_jA, \qquad a_i \equiv 0 \mod (m_iA+G)$$

for all $i, j \in \{1, \dots, k\}$. Conversely, let $a_i \equiv a_j \mod m_iA + m_jA$ and $a_i \equiv 0 \mod m_iA + G$ for all $i, j \in \{1, \dots, k\}$. The previous lemma yields a such that $a \equiv a_i \mod m_iA$ for $i = 1, \dots, k$. Hence by (4.1.2),

$$a \in \bigcap_{i=1}^{k}(m_iA+G) = \big(\bigcap_{i=1}^{k} m_iA\big) + G.$$

So $a = a' + g$ with $a' \in \bigcap_{i=1}^{k} m_iA$. Thus $g \equiv a_i \mod m_iA$ for all i. ☐

In the rest of this section we assume A is *torsion-free*. Let $a_1, \dots, a_k$ and $g_1, \dots, g_l$ be given, and let $m_1, \dots, m_k, n_1, \dots, n_l > 0$ with $l \geq 2$. If

$$\big(\bigwedge_{i=1}^{k} g \equiv a_i \mod m_iA\big) \ \wedge \ \big(\bigwedge_{j=1}^{l} g \equiv g_j \mod n_jG\big), \tag{$\star$}$$

then $g - g_l = n_lg'$ and

$$\big(\bigwedge_{i=1}^{k} n_lg' \equiv a_i' \mod m_iA\big) \ \wedge \ \big(\bigwedge_{j=1}^{l-1} n_lg' \equiv g_j' \mod n_jG\big), \tag{$\star\star$}$$

where $a_i' = a_i - g_l$ and $g_j' = g_j - g_l$ for $i = 1, \dots, k$ and $j = 1, \dots, l-1$.

For $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, l-1\}$, let $\gamma_i = \gcd(m_i, n_l)$ and $\delta_j = \gcd(n_j, n_l)$ and fix integers x_i, y_i, w_j, z_j such that $x_im_i + y_in_l = \gamma_i$ and $w_jn_j + z_jn_l = \delta_j$. Then the condition ($\star\star$) on arbitrary g' is equivalent to

$$\begin{aligned}&\big(\bigwedge_{i=1}^{k} a_i' \in \gamma_i A \ \wedge \ \frac{n_l}{\gamma_i} g' \equiv \frac{a_i'}{\gamma_i} \mod \frac{m_i}{\gamma_i} A\big)\\ \wedge\ &\big(\bigwedge_{j=1}^{l-1} g_j' \in \delta_j G \ \wedge \ \frac{n_l}{\delta_j} g' \equiv \frac{g_j'}{\delta_j} \mod \frac{n_j}{\delta_j} G\big).\end{aligned}$$

For $i = 1, \ldots, k$ the integers y_i and m_i/γ_i are coprime, so the condition

$$a_i' \in \gamma_i A \ \wedge \ (n_l/\gamma_i) g' \equiv a_i'/\gamma_i \mod (m_i/\gamma_i) A$$

on g' is equivalent to

$$a_i' \in \gamma_i A \ \wedge \ g' \equiv y_i(a_i'/\gamma_i) \mod (m_i/\gamma_i) A.$$

Likewise, for $j = 1, \ldots, l-1$ the condition

$$g_j' \in \delta_j G \ \wedge \ (n_l/\delta_j) g' \equiv g_j'/\delta_j \mod (n_j/\delta_j) G$$

on g' is equivalent to

$$g_j' \in \delta_j G \ \wedge \ g' \equiv z_j(g_j'/\delta_j) \mod (n_j/\delta_j) G.$$

Hence the condition $(\star\star)$ on g' is equivalent to

$$\begin{aligned}&\big(\bigwedge_{i=1}^{k} a_i' \in \gamma_i A \ \wedge \ g' \equiv y_i \frac{a_i'}{\gamma_i} \mod \frac{m_i}{\gamma_i} A\big)\\ \wedge\ &\big(\bigwedge_{j=1}^{l-1} g_j' \in \delta_j G \ \wedge \ g' \equiv z_j \frac{g_j'}{\delta_j} \mod \frac{n_j}{\delta_j} G\big).\end{aligned}$$

If g satisfies $(\star)$, then iterating this process we get $a_1^*, \ldots, a_k^*$ and integers $m_1^*, \ldots, m_k^*, n > 0$ depending only on

$$a_1, \ldots, a_k, \ g_1, \ldots, g_l, \ m_1, \ldots, m_k, \ n_1, \ldots, n_l$$

and a g^* depending *also* on g such that

$$\big(\bigwedge_{i=1}^{k} g^* \equiv a_i^* \mod m_i^* A\big) \wedge \big(g^* \equiv 0 \mod nG\big). \tag{$\star\star\star$}$$

Moreover, this process can be reversed to give g as in $(\star)$ from g^* as in $(\star\star\star)$. Therefore, by Lemma 4.1.3

with nG in the place of G, we get

Lemma 4.1.4. *Let $a_1, \ldots, a_k$, $g_1, \ldots, g_l$, and $m_1, \ldots m_k, n_1, \ldots, n_l > 0$ be given, with $l \geq 2$. Then there are $a_1^*, \ldots, a_k^*$ and $m_1^*, \ldots, m_k^*, n > 0$ such that the following are equivalent :*

(i) there is g satisfying $(\star)$,

(ii) for all $i, j \in \{1, \ldots, k\}$, $a_i^* \equiv a_j^* \mod m_i^* A + m_j^* A$ and $a_i^* \equiv 0 \mod m_i^* A + nG$.

Remark. Assume also that $a_1, \ldots, a_k$ lie in a pure subgroup A' of A, and $g_1, \ldots, g_l \in G' := G \cap A'$. Then G' is pure in G, and the proof shows that the conclusion of the lemma holds with $a_1^*, \ldots, a_k^* \in A'$ and $m_1^*, \ldots, m_k^*, n$ depending only on

$$(A', G', a_1, \ldots, a_k, m_1, \ldots, m_k, g_1, \ldots, g_l, n_1, \ldots, n_l),$$

not on the ambient structure (A, G). The purity of A' in A also shows that in (ii) one can replace A and G by A' and G'. It follows that if there is a g as in (i), then there is already such a g in G'.

For each p, let $e(p), f(p) \in \mathbb{N}_\infty$ be such that $p^{e(p)} = [p]A$ and $p^{f(p)} = [p]G$. Put $e := \big(e(p)\big)$ and $f := \big(f(p)\big)$ and call e, f the *system of prime invariants of* (A, G). Note that if $a_1, \ldots, a_n \in A$ are $\mathbb{Z}$-linearly dependent, then $a_1/pA, \ldots, a_n/pA$ are linearly dependent in the $\mathbb{F}_p$-linear space A/pA. In particular, if A has finite rank, then for every p,

$$e(p) = \dim_{\mathbb{F}_p} A/pA \leq \operatorname{rk} A < \infty.$$

In the rest of this section we assume that A/pA and G/pG are finite for each p. Hence for $n > 0$, the group A/nA is finite, and

$$[n]A = |A/nA| = [p_1]A \cdots [p_k]A$$

if $n = p_1 \cdots p_k$. Likewise, $[n]G = |G/nG| < \infty$ for $n > 0$.

4.2 Model theory of pairs of torsion-free abelian groups

Let $\mathcal{L}_{\text{ab}} := \{+, -, 0\}$ be the language of abelian groups, and let $\mathcal{L}_{\text{ab}}(V)$ extend $\mathcal{L}_{\text{ab}}$ by a unary predicate symbol V. Take A and G as in the previous section; that is A is a torsion-free abelian group, G is a subgroup of A such that $[p]A$ and $[p]G$ are finite for each p.

Now let A' be a subgroup of A such that A' is pure in A and

$$[p]A' \geq [p]A,$$

for each p. These assumptions yield that for each $n > 0$ the natural group morphism

$$A'/nA' \to A/nA$$

is an isomorphism. Let $G' := G \cap A'$. Then G' is pure in G. Assume also that $[p]G' \geq [p]G$ for every p. Again, the natural group morphism

$$G'/nG' \to G/nG$$

is an isomorphism for every $n > 0$.

Now let B be a torsion-free abelian group with subgroup H such that (B, H) has the same system of prime invariants as (A, G). Let (B', H') be an $\mathcal{L}_{\mathrm{ab}}(V)$-substructure of (B, H) such that $|B'| = |A'|$, B' is pure in B and such that $[p]B' \geq [p]B$, and $[p]H' \geq [p]H$ for each p. Assume that (A, G) and (B, H) are κ-saturated for an infinite cardinal $\kappa > |A'| = |B'|$, and let

$$\iota : (A', G') \to (B', H')$$

be an isomorphism of $\mathcal{L}_{\mathrm{ab}}(V)$-structures, that is an isomorphism of groups A' and B' such that $\iota(G') = H'$.

Lemma 4.2.1. *Let g be given. Then there is $h \in H$ such that for all $a' \in A'$, $g' \in G'$, $m, n > 0$, and $k, l \in \mathbb{Z}$,*

$$kg \equiv a' \mod mA \iff kh \equiv \iota(a') \mod mB, \tag{4.2.1}$$

$$lg \equiv g' \mod nG \iff lh \equiv \iota(g') \mod nH. \tag{4.2.2}$$

Proof. Note that if $h \in B$ satisfies (4.2.2) for $g' = 0$ and $l = n = 1$, then $h \in H$, since $g \in G$. So we can disregard the requirement that $h \in H$.

Claim. Let $h \in B$ be such that for all $a' \in A',\ g' \in G',\ m, n > 0$,

$$g \equiv a' \mod mA \iff h \equiv \iota(a') \mod mB, \tag{4.2.3}$$

$$g \equiv g' \mod nG \iff h \equiv \iota(g') \mod nH. \tag{4.2.4}$$

Then (4.2.1), and (4.2.2) hold for all $a' \in A',\ g' \in G',\ m, n > 0$, and $k, l \in \mathbb{Z}$.

Proof of the claim. Let $k \in \mathbb{Z}$ and $m > 0$. Let $\gamma := \gcd(k, m) = xk + ym$ with $x, y \in \mathbb{Z}$. Thus $1 = x\frac{k}{\gamma} + y\frac{m}{\gamma}$, so x and $\frac{m}{\gamma}$ are coprime, hence

$$\begin{aligned} kg - a' \in mA &\iff a' \in \gamma A \text{ and } \frac{k}{\gamma}g - \frac{a'}{\gamma} \in \frac{m}{\gamma}A \\ &\iff a' \in \gamma A \text{ and } g - \frac{xa'}{\gamma} \in \frac{m}{\gamma}A. \end{aligned}$$

Likewise, if $l \in \mathbb{Z}$, $n > 0$, and $\delta := \gcd(l, n) = wl + zn$ with $w, z \in \mathbb{Z}$, then

$$lg - g' \in nG \iff g' \in \delta G \text{ and } g - \frac{wg'}{\delta} \in \frac{n}{\delta}G.$$

So the claim follows by the purity of A' in A and G' in G.

Let $a'_1, \dots, a'_s \in A'$, $g'_1, \dots, g'_t \in G'$, and let $m_1, \dots, m_s, n_1, \dots, n_t > 0$ be such that $g \equiv a'_i \mod m_iA$ for $i = 1, \dots, q$, $g \not\equiv a'_i \mod m_iA$ for $i = q+1, \dots, s$, $g \equiv g'_j \mod n_jG$ for $j = 1, \dots, r$, and $g \not\equiv g'_j \mod n_jG$ for $j = r+1, \dots, t$. By saturation it suffices to find $b \in B$ such that $b \equiv \iota(a'_i) \mod m_iB$ for $i = 1, \dots, q$, $b \not\equiv \iota(a'_i) \mod m_iB$ for $i = q+1, \dots, s$, $b \equiv \iota(g'_j) \mod n_jH$ for $j = 1, \dots, r$, and $b \not\equiv \iota(g'_j) \mod n_jH$ for $j = r+1, \dots t$.

By the isomorphisms $A'/mA' \simeq A/mA$ and $G'/nG' \simeq G/nG$ for $m, n > 0$ we have $g \equiv a'_i + c'_i \mod m_iA$ and $g \equiv g'_j + d'_j \mod n_jG$ for $i = q+1, \dots, s$, and $j = r+1, \dots, t$, where $c'_{q+1}, \dots, c'_s \in A'$, and $d'_{r+1}, \dots, d'_t \in G'$. Note that then $c'_i \notin m_iA$ for $i = q+1, \dots, s$ and $d'_j \notin n_jG$ for $j = r+1, \dots, t$. Hence, if $b \in B$ and $b \equiv \iota(a'_i + c'_i) \mod m_iB$ for $i = q+1, \dots, s$ and $b \equiv \iota(g'_j + d'_j) \mod n_jH$ for $j = r+1, \dots, t$, then $b \not\equiv \iota(a'_i) \mod m_iB$ for $i = q+1, \dots, s$, and $b \not\equiv \iota(g'_j) \mod n_jH$ for $j = r+1, \dots, t$. Therefore, replacing a'_i by $a'_i + c'_i$ for $i > q$ and g'_j by $g'_j + d'_j$ for $j > r$, we reduce to the case that $q = s$ and $r = t$ with the incongruences replaced by congruences.

Lemma 4.1.4 and the subsequent remark show that these $s + t$ congruences satisfied by g are satisfied by some $g' \in G'$, and then $b := \iota(g') \in B'$ satisfies the corresponding congruences in (B, H), using that B' is pure in B. $\square$

Now let $\mathcal{I}$ be the set of $\mathcal{L}_{\text{ab}}$-isomorphisms

$$\iota : (A', G') \to (B', H'),$$

with A', G', B' and H' as above; that is A' and B' are pure subgroups of A and B respectively, $|A'| < \kappa$, $|B'| < \kappa$, $G' = A' \cap G$, $H' = B' \cap H$ and $[p]A' \geq [p]A$, $[p]G' \geq [p]G$, $[p]B' \geq [p]B$, and $[p]H' \geq [p]H$ for every p. As noted above, for such A', G', B', H', we have that G' and H' are pure subgroups of G and H

respectively, and that $[p]A' = [p]A$, $[p]G' = [p]G$, $[p]B' = [p]B$, and $[p]H' = [p]H$ for every p. Moreover the finite groups A'/pA' and A/pA are isomorphic via the natural inclusion map. Likewise for every p, B'/pB', G'/pG', and H'/pH' are isomorphic to B/pB, G/pG, and H/pH respectively.

Lemma 4.2.2. *The collection $\mathcal{I}$ is a back-and-forth system.*

Proof. Let $\iota : (A', G') \to (B', H')$ be in $\mathcal{I}$ and $a \in A \setminus A'$.

First let $a \in G$. Using 4.2.1, we take $b \in H$ such that for all $a' \in A'$, $g' \in G'$, $m, n >$ and $k, l \in \mathbb{Z}$,

$$kg \equiv a' \mod mA \iff kh \equiv \iota(a') \mod mB,$$

$$lg \equiv g' \mod nG \iff lh \equiv \iota(g') \mod nH.$$

Now the map $\frac{a'+kg}{n} \mapsto \frac{\iota(a')+kh}{n} : A'\langle a\rangle_A \to B'\langle b\rangle_B$ is a group morphism, sending $G'\langle a\rangle_G$ to $H'\langle b\rangle_H$. Hence it is in $\mathcal{I}$.

Now let $a \in A'\langle G\rangle_A$. Then $a = \frac{a'+kg}{n}$ with $a' \in A'$, $g \in G$, $k \in \mathbb{Z}$ and $n > 0$. Now using the previous case we get $h \in H$ such that ι extends to $(A'\langle g\rangle_A, G'\langle g\rangle_G) \to (B'\langle h\rangle_B, H\langle h\rangle_H)$, which is in $\mathcal{I}$. We are done with this case as $a \in A'\langle g\rangle_A$.

Finally let $a \notin A'\langle G\rangle_A$. Note that by saturation $B'\langle H\rangle_B \neq B$ is nonempty. Take $b \in B \setminus B'\langle H\rangle_B$. Then we have a group morphism $A'\langle a\rangle_A \to B'\langle b\rangle_B$ taking a to b. It is indeed an isomorphism of $\mathcal{L}_{\mathrm{ab}}(V)$-structures $(A'\langle a\rangle_A, G')$ and $(B'\langle b\rangle_B, H')$ since $A'\langle a\rangle_A \cap G = G'$ and $B'\langle b\rangle_B \cap H = H'$. It is easy to check that this isomorphism is in $\mathcal{I}$. ☐

Corollary 4.2.3. *Let $(A', G') \subseteq (A, G)$ where A' is a pure subgroup of A, and (A', G') and (A, G) have the same prime invariants. Then $(A', G') \preceq (A, G)$.*

Proof. We may assume that (A, G) is $|A'|^+$-saturated. Also take another $|A'|^+$-saturated elementary extension, (B, H), of (A', G'). Then there is a back-and-forth system between (A, G) and (B, H) as constructed above, which contains the identity map on (A', G'). Hence $(A', G') \preceq (A, G)$. ☐

Let $e = \big(e(p)\big)$, and $f = \big(f(p)\big)$ be two families of natural numbers indexed by the prime numbers. Let $T(e, f)$ be the $\mathcal{L}_{\mathrm{ab}}(V)$-theory whose models are of the form (A, G), where A is a torsion-free abelian group and G is a subgroup of A such that the system of prime invariants of (A, G) is e, f.

Next we prove a model completeness result after enriching the language. Let $\mathcal{L}_{\mathrm{ab}}(V)^*$ be the language augmenting $\mathcal{L}_{\mathrm{ab}}(V)$ by a unary predicate D_p for every p, and let $T^*(e, f)$ be the $\mathcal{L}_{\mathrm{ab}}(V)^*$-theory extending

$T(e,f)$ by the following defining axiom for D_p:

$$\forall x\big(D_p(x) \leftrightarrow \exists y(x = py)\big). \tag{4.2.5}$$

A model (A,G) of $T(e,f)$ becomes a model of $T^*(e,f)$ by interpreting D_p as pA.

Corollary 4.2.4. *$T^*(e,f)$ is model complete.*

Proof. Let $(A',G') \subseteq (A,G)$ be two models of $T^*(e,f)$. Then A' is pure in A. Thus using Corollary 4.2.3 we get $(A',G') \preceq (A,G)$. $\square$

4.3 A model completeness result for abelian groups

Here we consider abelian groups rather than pairs of abelian groups as we did in the previous sections.

Let A, B be (additively written) abelian groups with $A \subseteq B$. It is obvious that if A is existentially closed in B, then A is pure in B. Proposition 4.3.2 states that the converse holds under a mild assumption.

Lemma 4.3.1. *Suppose that $mA \neq \{0\}$ for all $m > 0$. Let E be a finite subset of A and let $n > 0$. Then there is $a \in A$ such that $na \notin E$.*

Proof. Suppose that $nA \subseteq E$. Then nA is a finite group, say of size m, and then $mnA = \{0\}$, contradicting the assumption on A. $\square$

Proposition 4.3.2. *Suppose A is pure in B, every torsion element of B lies in A, and $mA \neq \{0\}$ for all $m > 0$. Then A is existentially closed in B.*

Proof. We may assume that B is finitely generated over A. It follows from the first two assumptions that then $B = A \oplus \mathbb{Z}b_1 \oplus \cdots \oplus \mathbb{Z}b_n$, with $b_1, \ldots, b_n \in B$. By induction on n we reduce to the case $n = 1$, so $B = A \oplus \mathbb{Z}b$ with $b \in B$. The previous lemma yields an elementary extension A^* of A with an element $a^* \in A^*$ such that $na^* \notin A$ for every $n > 0$. Then the inclusion $A \to A^*$ extends to embedding $B \to A^*$ that sends b to a^*. $\square$

Let A_0 be an abelian group such that A_0/nA_0 and $A_0[n]$ are finite for every $n > 0$. Let $\mathcal{L}_0$ be the language of abelian groups augmented by names for elements of A_0, and let T_0 be the $\mathcal{L}_0$-theory of the class of abelian groups A such that A_0 is a pure subgroup of A, and for all $n > 0$ we have $[n]A = [n]A_0$, $A[n] = A_0[n]$, and $nA \neq \{0\}$.

Proposition 4.3.3. *The theory T_0 is model-complete.*

Proof. Let A' and A be models of T_0 with $A_0 \subseteq A' \subseteq A$. By Proposition 4.3.2 it suffices to show that A' is pure in A. Let $n > 0$, $a' \in A'$ and $a \in A$ with $na = a'$. Take $a_0 \in A_0$ and $a'' \in A'$ such that $a' = a_0 + na''$. Then $n(a - a'') = a_0$, and hence by the purity of A_0 in A and $A_0[n] = A[n]$, we get $a - a'' \in A_0$. So $a \in A'$. Therefore A' is pure in A. $\square$

Chapter 5

Pairs of ordered abelian groups

Here we consider pairs of ordered abelian groups. In Chapter 6 we apply the results of the current chapter to structures $(\mathbb{R}, A, G)$ where $\mathbb{R}$ is the field of real numbers and A and $G \subseteq A$ are subgroups of the multiplicative group $\mathbb{R}^{>0}$. As mentioned in the Introduction, our main interest is in the case that A/G is infinite.

Later we use multiplicative notation in dealing with subgroups of multiplicative groups of fields, but in the current chapter additive notation is more convenient.

5.1 Pairs of regularly dense ordered abelian groups

Let A be an ordered abelian group; so A is in particular torsion-free. We recall here some terminology from [22]. We say that A is *regularly dense* if A is non-trivial and for every p and all $a < b$ in A we have $pA \cap (a, b) \neq \emptyset$.

The reason for introducing this notion is its first-order nature and the fact that ordered subgroups of the multiplicative group $\mathbb{R}^{>0}$ that are dense in $\mathbb{R}^{>0}$ are regularly dense.

If A is regularly dense, then A has no smallest positive element, so A, as a linearly ordered set, is dense without endpoints.

Note that if A is regularly dense, G is an ordered subgroup of A and G is dense in A, then G is regularly dense.

In the rest of this section A is a regularly dense ordered abelian group, G is an ordered subgroup of A such that G is dense in A, A/G is infinite, and $[p]A$ and $[p]G$ are finite for each p. (These assumptions are clearly satisfied if A is a dense subgroup of $\mathbb{R}^{>0}$ of finite rank and G is a subgroup of A such that G is dense in A and A/G is infinite.)

Let $\mathcal{L}_{\mathrm{oab}} = \{<, 0, +, -\}$ be the language of ordered abelian groups, and $\mathcal{L}_{\mathrm{oab}}(V)$ the language extending $\mathcal{L}_{\mathrm{oab}}$ by a unary predicate symbol V.

Consider a second pair (B, H) where B is a regularly dense ordered abelian group and H is an ordered subgroup of B such that H is dense in B, and B/H is infinite. Assume also that (A, G) and (B, H) have the same system of prime invariants and are κ-saturated where κ is an uncountable cardinal. Let $\mathcal{I}$ be the set of isomorphisms $\iota : (A', G') \to (B', H')$ between $\mathcal{L}_{\text{oab}}(V)$-substructures (A', G') of (A, G) and (B', H') of (B, H) of cardinality less than κ, such that:

(1) A', B' are pure subgroups of A, B, respectively;

(2) for each p,

$$[p]A' \geq [p]A, \quad [p]G' \geq [p]G, \quad [p]B' \geq [p]B, \quad [p]H' \geq [p]H.$$

If $\iota : (A', G') \to (B', H')$ is in $\mathcal{I}$, then $G' = A' \cap G$ and $H' = B' \cap H$ are pure subgroups of G and H respectively; moreover for each p

$$[p]A' = [p]A, \quad [p]G' = [p]G, \quad [p]B' = [p]B, \quad [p]H' = [p]H.$$

We proceed to show that $\mathcal{I}$ has the back-and-forth property. So let

$$\iota : (A', G') \to (B', H')$$

be in $\mathcal{I}$, and $a \in A \setminus A'$. We want to extend ι to an $\mathcal{L}_{\text{oab}}(V)$-isomorphism in $\mathcal{I}$ whose domain contains a.

First consider the case $a \in G$. Then by Lemma 4.2.1 we can take $b \in H$ such that for all $a' \in A'$, $g' \in G'$, $m, n > 0$ and $k, l \in \mathbb{Z}$:

$$a' + ka \in mA \Longleftrightarrow \iota(a') + kb \in mB,$$

$$g' + la \in nG \Longleftrightarrow \iota(g') + lb \in nH.$$

We wish to modify b so that it still satisfies the above and that for each $N \in \mathbb{N}^{>0}$, Nb realizes the cut in B' corresponding to the cut of Na in A' via ι. By saturation, it suffices to modify b so that for given $N_1, \ldots, N_s \in \mathbb{N}^{>0}$, $N_1 b, \ldots, N_s b$ realize the cuts in B' corresponding to the cuts of $N_1 a, \ldots, N_s a$ in A' via ι, respectively. Moreover, taking $N := N_1 \cdots N_s$, we reduce to the case that $s = 1$.

So let $N \in \mathbb{N}^{>0}$ be given. Using saturation and the density of H in B we pick $b_1 \in H$ so that b_1 realizes the cut in B' corresponding to the cut of Na in A'. Take $\varepsilon \in H$ such that $0 < \varepsilon < b'$ for all $b' \in B'$; so all elements of $[b_1, b_1 + \varepsilon] \subseteq H$ realize the same cut in B' as b_1. As H is regularly dense, saturation yields $h \in [b_1 - Nb, b_1 + \varepsilon - Nb] \cap \bigcap_{n>0} nH$. Then $Nb + h$ realizes the cut in B' corresponding to the cut of Na in A' via ι. It follows that $b + \frac{h}{N}$ is a modification of b as desired.

We can now assume that for each $N \in \mathbb{N}^{>0}$, Nb realizes the cut in B' corresponding to the cut of Na in A'. This allows us to extend ι to an $\mathcal{L}_{\text{oab}}(V)$-isomorphism

$$\big(A'\langle a\rangle_A, G'\langle a\rangle_G\big) \to \big(B'\langle b\rangle_B, H'\langle b\rangle_H\big), \qquad a \mapsto b.$$

Note that $A'\langle a\rangle_A$ and $B'\langle b\rangle_B$ are pure subgroups of A and B respectively, and that $A'\langle a\rangle_A \cap G = G'\langle a\rangle_G$ and $B'\langle b\rangle_B \cap H = H'\langle b\rangle_H$. It remains to check that (2) holds with $A'\langle a\rangle_A$, $G'\langle a\rangle_G$, $B'\langle b\rangle_B$, $H'\langle b\rangle_H$ instead of A', G', B', H'. Let any p be given. By purity we have

$$[p]A'\langle a\rangle_A \geq [p]A',\ [p]G'\langle a\rangle_G \geq [p]G',\ [p]B'\langle b\rangle_B \geq [p]B,\ [p]H'\langle b\rangle_G \geq [p]H',$$

which in view of (2) yields the desired inequalities. Therefore the above extension of ι is in $\mathcal{I}$.

Next, consider the case that $a \in A'\langle G\rangle_A$. Then $a = (a' + g)/m$, where $a' \in A'$, $g \in G$, and $m > 0$ with $a' + g \in mA$. Then by the previous case, we can extend ι to an $\mathcal{L}_{\text{oab}}(V)$-isomorphism in $\mathcal{I}$ with g in its domain, and thus a in its domain as well.

Finally, assume that $a \in A \setminus A'\langle G\rangle_A$. Since B/H is infinite, it follows by saturation that $B'\langle H\rangle_B \neq B$ and moreover that $B \setminus B'\langle H\rangle_B$ is dense in B. As in the first case we get $b \in B \setminus B'\langle H\rangle_B$ such that for each $N \in \mathbb{N}^{>0}$, Nb realizes the cut in B' corresponding via ι to the cut that Na realizes in A', and for all $a' \in A'$, $n > 0$, and $k \in \mathbb{Z}$

$$a' + ka \in mA \Longleftrightarrow \iota(a') + kb \in mB.$$

Hence ι extends to an $\mathcal{L}_{\text{oab}}(V)$-isomorphism

$$\big(A'\langle a\rangle_A, G'\big) \to \big(B'\langle b\rangle_B, H'\big), \quad a \mapsto b.$$

By the same argument as in the first case it follows that this extension of ι is in $\mathcal{I}$. This finishes the proof of the fact that $\mathcal{I}$ has the back-and-forth property.

Corollary 5.1.1. *Let (A', G') be a substructure of (A, G) such that A' is pure in A, and regularly dense, and G' is dense in A'. Suppose that A'/G' is infinite and (A, G) and (A', G') have the same system of prime invariants. Then $(A', G') \preceq (A, G)$.*

Proof. Let κ be an uncountable cardinal bigger than $|A'|$, and take a κ-saturated elementary extension (A^*, G^*) of (A', G'). We may assume that (A, G) is κ-saturated as well. So there is a back-and-forth system $\mathcal{I}$ between (A, G) and (A^*, G^*), containing the identity map on (A', G'). This yields the desired result. $\square$

Let $e = \big(e(p)\big)$, and $f = \big(f(p)\big)$ be two families of natural numbers indexed by the prime numbers. Let $T_{\mathrm{o}}(e,f)$ be the $\mathcal{L}_{\mathrm{oab}}(V)$-theory whose models are the pairs (A,G), where A is a regularly dense ordered abelian group, G is an ordered subgroup of A such that G is dense in A, A/G is infinite, and the system of prime invariants of (A,G) is e,f. To show that $T_{\mathrm{o}}(e,f)$ has a model, let $\mathbb{Z}_{(p)}$ be the additive group of the localization of $\mathbb{Z}$ at the prime ideal $p\mathbb{Z}$, viewed as a subgroup of the additive group of $\mathbb{Q}$; in other words $\mathbb{Z}_{(p)} = \{\frac{x}{y} \in \mathbb{Q} : x, y \in \mathbb{Z}, p \not| y\}$. Note that $[p]\mathbb{Z}_{(p)} = p$ and $[p']\mathbb{Z}_{(p)} = 1$ for every prime number $p' \neq p$.

First assume that f is not identically 0. Put

$$A := \bigoplus_p \mathbb{Z}_{(p)}^{e(p)} \oplus \bigoplus_p \mathbb{Q}^{f(p)}, \text{ and } \; G := \bigoplus_p \mathbb{Z}_{(p)}^{f(p)},$$

Consider G as a subgroup of A by identifying G with a subgroup of the second summand $\bigoplus_p \mathbb{Q}^{f(p)}$ of A in the obvious way. Note that the system of prime invariants of (A,G) is e,f. Since $\mathbb{R}$ is an infinite dimensional $\mathbb{Q}$-vector space, we can embed A as an abelian group into the additive group of $\mathbb{R}$. Such an embedding makes A and G into regularly dense ordered abelian groups with the order induced from $\mathbb{R}$, and then $(A,G) \models T_{\mathrm{o}}(e,f)$.

Next, let $f(p) = 0$ for every p, and let e be not identically 0. Put

$$A = \mathbb{Q} \oplus \bigoplus_p \mathbb{Z}_{(p)}^{e(p)}, \text{ and}$$

$$G = \mathbb{Q}.$$

Consider G as the first summand in the above direct sum decomposition of A. As in the previous paragraph we can embed A as an abelian group into the additive group of $\mathbb{R}$; such an embedding makes A and G into regularly dense ordered abelian groups with the order induced from $\mathbb{R}$, and then $(A,G) \models T_{\mathrm{o}}(e,f)$.

Finally, let $e(p) = f(p) = 0$ for each p. Let α be an irrational real number, and let $A := \mathbb{Q} + \mathbb{Q}\alpha \subseteq \mathbb{R}$, and $G := \mathbb{Q} \subseteq \mathbb{R}$. Then (A,G) equipped with the order induced from $\mathbb{R}$ is a model of $T_{\mathrm{o}}(e,f)$.

For each n we put $\mathbf{n} := (n_p)$ with $n_p = n$ for all p.

Next we prove a quantifier eliminability result for $T_{\mathrm{o}}(\mathbf{0}, f)$. Fix a tuple of distinct variables $x = (x_1, \ldots, x_m)$. A *special* $\mathcal{L}_{\mathrm{oab}}(V)$-formula in x is a formula, $\psi(x)$, of the form

$$\exists y \big(V(y) \wedge \theta_V(y) \wedge \phi(x,y)\big),$$

where $y = (y_1, \dots, y_n)$ is a tuple of distinct variables, $\theta(y)$ and $\phi(x, y)$ are $\mathcal{L}_{\mathrm{oab}}$-formulas, and $\theta_V(y)$ is the V-restriction of $\theta(y)$ as defined in Chapter 2.

Lemma 5.1.2. *Each $\mathcal{L}_{\mathrm{oab}}(V)$-formula $\psi(x)$ is equivalent in $T_{\mathrm{o}}(\mathbf{0}, f)$ to a boolean combination of special $\mathcal{L}_{\mathrm{oab}}(V)$-formulas in x.*

Proof. Let (A, G) and (B, H) be two $\aleph_1$-saturated models of $T_{\mathrm{o}}(\mathbf{0}, f)$, and let $\mathcal{I}$ be the back-and-forth system between (A, G) and (B, H) as constructed earlier in this subsection (taking $\kappa = \aleph_1$). Now let $a \in A^m$ and $b \in B^m$ satisfy the same special formulas in A and B respectively. It suffices to prove that a and b satisfy the same types in (A, G) and (B, H) respectively. To show this, we find an element ι of $\mathcal{I}$ with $\iota(a) = b$.

Let $\mathrm{rk}(G\langle a\rangle_A/G) = r$. We may assume that

$$\mathrm{rk}(G\langle a_1, \dots, a_r\rangle_A/G) = r, \text{ and } a_i \in G\langle a_1, \dots, a_r\rangle_A$$

for $r < i \leq m$. Then since a and b satisfy the same special formulas, we have $\mathrm{rk}(H\langle b_1, \dots, b_r\rangle_B/H) = r$ and $b_i \in H\langle b_1, \dots, b_r\rangle_B$ for $r < i \leq m$.

Now let G' be a countable pure subgroup of G such that

$$\mathrm{rk}(G'\langle a\rangle_A/G') = r, \text{ and } [p]G' = [p]G$$

for each p. Note that $G'\langle a\rangle_A$ is divisible, since it is a pure subgroup of the divisible group A. Thus $[p]G'\langle a\rangle_A = [p]A = 1$ for every p. Also it is easy to check that $G'\langle a\rangle_A \cap G = G'$. Hence $(G'\langle a\rangle_A, G')$ is a countable substructure of (A, G).

Enumerate G' as $g = (g_0, g_1, \dots)$, and let $y = (y_0, y_1, \dots)$ be a countable tuple of distinct variables. If $\theta_1(y), \dots, \theta_k(y)$ and $\phi_1(x, y), \dots, \phi_k(x, y)$ are $\mathcal{L}_{\mathrm{oab}}$-formulas such that $G \models \theta_i(g)$ and $A \models \phi_i(a, g)$ for $i = 1, \dots, k$, then

$$(A, G) \models \exists y\big(V(y) \wedge \theta_V(y) \wedge \phi(a, y)\big),$$

where $\theta(y) := \bigwedge_{i=1}^k \theta_i(y)$, and $\phi(x, y) := \bigwedge_{i=1}^k \phi_i(x, y)$. Therefore

$$(B, H) \models \exists y\big(V(y) \wedge \theta_V(y) \wedge \phi(b, y)\big).$$

So we have a partial y-type over b in (B, H), consisting of formulas $\theta_V(y)$ and $\phi(b, y)$ where $G \models \theta(g)$ and $A \models \phi(a, g)$. Thus by saturation there is a countable tuple $h = (h_0, h_1, \dots)$ realizing this type.

Now let $H' = \{h_0, h_1, \dots\}$. Then H' is a countable pure subgroup of H which is $\mathcal{L}_{\mathrm{oab}}$-isomorphic to

G' such that $\mathrm{rk}(H'\langle b\rangle_B/H') = r$, $[p]H' = [p]H$ for every p, and $H'\langle b\rangle_B$ is divisible. Moreover there is an ordered group isomorphism $G'\langle a\rangle_A \to H'\langle b\rangle_B$ sending g_i to h_i for $i = 0, 1 \ldots$, and a to b. It is easy to check that (1) and (2) are satisfied with $G'\langle a\rangle_A, H'\langle b\rangle_B$ in the place of A', B'. Hence the $\mathcal{L}_{\mathrm{oab}}(V)$-isomorphism

$$\iota : \big(G'\langle a\rangle_A, G'\big) \to \big(H'\langle b\rangle_B, H'\big)$$

is in $\mathcal{I}$. $\square$

Also as a consequence of Lemma 5.1.2, we get

Corollary 5.1.3. *The $\mathcal{L}_{\mathrm{oab}}(V)$-theory $T_{\mathrm{o}}(\mathbf{0}, f)$ is complete.*

Proof. Let σ be an $\mathcal{L}_{\mathrm{oab}}(V)$-sentence. Then by Lemma 5.1.2, we may assume that σ is of the form

$$\exists y\big(V(y) \wedge \theta_V(y) \wedge \phi(y)\big),$$

where $\theta(y)$ and $\phi(y)$ are $\mathcal{L}_{\mathrm{oab}}$-formulas. Moreover we may choose $\phi(y)$ to be quantifier-free since the theory of divisible ordered abelian groups admits quantifier elimination. Thus σ holds in a model (A, G) of $T_{\mathrm{o}}(\mathbf{0}, f)$ if and only if $\exists y(\theta(y) \wedge \phi(y))$ holds in G. By [22], the theory of regularly dense ordered abelian groups whose system of prime invariants is f is complete. Thus the completeness of $T_{\mathrm{o}}(\mathbf{0}, f)$ follows. $\square$

The next example illustrates that the completeness result above cannot be generalized to $T_{\mathrm{o}}(e, f)$ for arbitrary e, f.

Example. Let $\alpha, \beta \in \mathbb{R}$ be algebraically independent over $\mathbb{Q}$. Consider two models $(\mathbb{Z}+\mathbb{Q}\alpha+\mathbb{Q}\beta, \mathbb{Z}+\mathbb{Q}\alpha)$ and $(\mathbb{Q}+\mathbb{Q}\alpha+\mathbb{Z}\beta, \mathbb{Z}+\mathbb{Q}\alpha)$ of $T_{\mathrm{o}}(\mathbf{1}, \mathbf{1})$. Let $m > 0$ and define σ_m to be the sentence $\forall x(V(x) \to \exists y(x = my))$. Then for each $m > 0$

$$(\mathbb{Z}+\mathbb{Q}\alpha+\mathbb{Q}\beta, \mathbb{Z}+\mathbb{Q}\alpha) \models \neg\sigma_m$$

and

$$(\mathbb{Q}+\mathbb{Q}\alpha+\mathbb{Z}\beta, \mathbb{Z}+\mathbb{Q}\alpha) \models \sigma_m.$$

Thus $T_{\mathrm{o}}(\mathbf{1}, \mathbf{1})$ is not complete.

For $n > 0$, a similar example can be constructed to show that $T_{\mathrm{o}}(\mathbf{n}, \mathbf{n})$ is not complete.

Next we state a consequence of [22], as done on page 75 of [9]:

Lemma 5.1.4. *The theory of regularly dense ordered abelian groups whose system of prime invariants is* f *admits elimination of quantifiers after extending it by definitions as follows: augment* $\mathcal{L}_{\mathrm{oab}}$ *by extra unary predicates* D_d $(d > 0)$*, with defining axioms*

$$\forall x\bigl(D_d(x) \leftrightarrow \exists y(x = dy)\bigr).$$

Proposition 5.1.5. *Let* (A, G) *be a model of* $T_{\mathrm{o}}(\mathbf{0}, f)$*. Then every subset of* A^m *definable in* (A, G) *is a boolean combination of subsets of* A^m *defined in* (A, G) *by formulas* $\exists y\bigl(V(y) \wedge \psi(x, y)\bigr)$*, where* $\psi(x, y)$ *is a quantifier-free formula in the language* $\mathcal{L}_{\mathrm{oab}}$ *augmented by names for the elements of* A*.*

Proof. By Lemma 5.1.2 it is enough to prove that the subsets of A^m defined by special formulas are of the required form. So let $\theta(y)$ and $\phi(x, y)$ be $\mathcal{L}_{\mathrm{oab}}$-formulas. We find a quantifier-free $\mathcal{L}_{\mathrm{oab}}$-formula $\psi(x, z, t)$ and $g \in G^q$ with $z = (z_1, \ldots, z_p)$ and $t = (t_1, \ldots, t_q)$ tuples of distinct variables such that the formula $\exists y\bigl(V(y) \wedge \theta_V(y) \wedge \phi(x, y)\bigr)$ is equivalent in (A, G) to $\exists z\bigl(V(z) \wedge \psi(x, z, g)\bigr)$.

By Lemma 5.1.4, the set $\{g \in G^n : G \models \theta(g)\}$ is a boolean combination of subsets of G^n of the form

$$\{g \in G^n : k_1 g_1 + \cdots + k_n g_n = 0\},$$

$$\{g \in G^n : k_1 g_1 + \cdots + k_n g_n < 0\},$$

$$\{g \in G^n : k_1 g_1 + \cdots + k_n g_n \in dG\},$$

where $k_1, \ldots, k_n \in \mathbb{Z}$ and d is a positive integer.

Since we assume that $f(p)$ is a non-negative integer, the complement of the set

$$\{g \in G^n : k_1 g_1 + \cdots + k_n g_n \in dG\}$$

is a finite union of sets of the form

$$\{g \in G^n : k_1 g_1 + \cdots + k_n g_n \in g_0 + dG\},$$

where $g_0 \in G$.

Thus we get the desired $\psi(x, z)$, which can be chosen to be quantifier-free as the theory of divisible ordered abelian groups admits quantifier elimination. ☐

To get results analogous to Corollary 5.1.3 and Proposition 5.1.5 for arbitrary e, f, we augment the language $\mathcal{L}_{\mathrm{oab}}(V)$ by adding a unary predicate D_n for each $n > 0$; we call this new language $\mathcal{L}^*_{\mathrm{oab}}(V)$.

Now let $T^*_{\mathrm{o}}(e,f)$ be the $\mathcal{L}^*_{\mathrm{oab}}(V)$-theory extending $T_{\mathrm{o}}(e,f)$ by the following axiom for each $n > 0$,

$$\forall x\big(D_n(x) \leftrightarrow \exists y(x = ny)\big). \tag{5.1.1}$$

Proposition 5.1.6. *The $\mathcal{L}^*_{\mathrm{oab}}(V)$-theory $T^*_{\mathrm{o}}(e,f)$ is model complete.*

Proof. Let $(A', G') \subseteq (A, G)$ be two models of $T^*_{\mathrm{o}}(e,f)$. Note that by Lemma 5.1.4, we get that $A' \preceq A$. In particular A' is a pure subgroup of A. Then by using Corollary 5.1.1, we get $(A', G') \preceq (A, G)$. $\square$

Let $n > 0$. Write $n = p_1^{m_1} \cdots p_d^{m_d}$, and let

$$k(n) := p_1^{m_1 e(p_1)} \cdots p_d^{m_d e(p_d)} \text{ and } l(n) := p_1^{m_1 f(p_1)} \cdots p_d^{m_d f(p_d)}.$$

To get a quantifier elimination result, we augment $\mathcal{L}^*_{\mathrm{oab}}(V)$ further by adding a unary predicate E_n and $k(n) + l(n)$ many distinct constant symbols, $c_1^{(n)}, \ldots, c_{k(n)}^{(n)}, d_1^{(n)}, \ldots, d_{l(n)}^{(n)}$ for each $n > 0$; we call this new language $\mathcal{L}^{**}_{\mathrm{oab}}(V)$. Let $T^{**}_{\mathrm{o}}(e,f)$ be the $\mathcal{L}^{**}_{\mathrm{oab}}(V)$-theory extending $T^*_{\mathrm{o}}(e,f)$ by the following defining axioms for the new unary predicates and the new constant symbols (for every $n > 0$)

$$\forall x\big(E_n(x) \leftrightarrow V(x) \wedge \exists y(V(y) \wedge (x = ny))\big), \tag{5.1.2}$$

$$\Big(\bigwedge_{1 \le i < j \le k(n)} \neg D_n(c_i^{(n)} - c_j^{(n)})\Big) \wedge \forall x\Big(\bigvee_{i=1}^{k(n)} D_n(x - c_i^{(n)})\Big), \tag{5.1.3}$$

$$\big(\bigwedge_{i=1}^{l(n)} V(d_i^{(n)}) \wedge \bigwedge_{1 \le i < j \le l(n)} \neg E_n(d_i^{(n)} - d_j^{(n)})\big) \wedge \forall x\big(V(x) \to \bigvee_{i=1}^{l(n)} E_n(x - d_i^{(n)})\big). \tag{5.1.4}$$

Proposition 5.1.7. *The $\mathcal{L}^{**}_{\mathrm{oab}}(V)$-theory $T^{**}_{\mathrm{o}}(e,f)$ has quantifier elimination.*

Proof. We shall use Theorem 2.2.1. So let (A, G) be a countable model of $T^{**}_{\mathrm{o}}(e,f)$ with a proper $\mathcal{L}^{**}_{\mathrm{oab}}(V)$-substructure (A', G') and let (B, H) be an $\aleph_1$-saturated model of $T^{**}_{\mathrm{o}}(e,f)$, with an embedding ι of (A', G') into (B, H). We need to extend ι to an embedding of a substructure of (A, G), extending (A', G') properly, into (B, H).

We proceed as in the proof of the fact that $\mathcal{I}$ has the back-and-forth property, note however that here A' is not necessarily a pure subgroup of A, and (A, G) is not $\aleph_1$-saturated. Let $B' := \iota(A')$ and $H' := \iota(G')$.

First assume that there is $a' \in (A' \cap nA) \setminus nA'$ with $n > 0$. Choose n to be minimal such. Let $a \in A \setminus A'$ be such that $a' = na$. Thus $\iota(a') \in B' \cap nB$. Take $b \in B \setminus B'$ such that $\iota(a') = nb$. Note that this b realizes the cut in B' corresponding to the cut of a in A'. Now let $A'' := A' + \mathbb{Z}a$, $G'' := A'' \cap G$, and $B'' := B' + \mathbb{Z}b$. So A'' and B'' are isomorphic as ordered abelian groups, moreover this isomorphism takes G'' into H. Hence ι extends to an $\mathcal{L}^{**}_{\text{oab}}(V)$-embedding of (A'', G'') into (B, H) by taking a to b.

So we may assume that A' is pure in A. It follows then that $[p]A' = [p]A$ and $[p]G' = [p]G$ for any p. Let (A^*, G^*) be a $\aleph_1$-saturated elementary extension of (A, G). Thus there is back-and-forth system, $\mathcal{I}$, between (A^*, G^*) and (B, H), and $\iota : (A', G') \to (B', H')$ is in $\mathcal{I}$. Hence for $a \in A \setminus A'$, we can extend ι to a proper extension of (A', G') containing a, moreover this extension can be chosen to be included in (A, G). □

As a result of this proposition, we get:

Corollary 5.1.8. *The $\mathcal{L}_{\text{oab}}(V)$-theory $T_{\text{o}}(\mathbf{0}, \mathbf{0})$ admits quantifier elimination.*

Remark. Instead of adding a unary predicate symbol D_n for $n > 0$, we could have added a unary function symbol $\frac{1}{n}$ for each $n > 0$ to get $\mathcal{L}^*_{\text{oab}}(V)$; and we could have extended $T_{\text{o}}(e, f)$ to $T^*_{\text{o}}(e, f)$ by adding the following defining axiom

$$\forall x \forall y \big(\frac{1}{n}x = y \leftrightarrow (x = ny) \vee \forall z(x \neq nz \wedge y = 0)\big), \tag{5.1.5}$$

for each $n > 0$, instead of (5.1.1). We would still get the model completeness result 5.1.6. Also if we had extended $\mathcal{L}^*_{\text{oab}}(V)$ to $\mathcal{L}^{**}_{\text{oab}}(V)$ and $T^*_{\text{o}}(e, f)$ to $T^{**}_{\text{o}}(e, f)$ in a similar way, we would get the quantifier elimination result 5.1.7. Note that when we work with this new $\mathcal{L}^{**}_{\text{oab}}(V)$, the meaning of substructure changes. Indeed if (A', G') is an $\mathcal{L}^{**}_{\text{oab}}(V)$-substructure of (A, G), then A' is a pure subgroup of A.

5.1.1 The induced structure on A/G

Let (A, G) be a model of $T_{\text{o}}(e, f)$. We are interested in the structure induced on the definable quotient A/G by (A, G). More precisely, we define the structure $(A/G)_{\text{ind}}$ to have underlying set A/G, and to have as its 0-definable n-ary relations the sets $\pi^n(Y)$, where

$$\pi : A \to A/G$$

is the usual projection map and $Y \subseteq A^n$ is definable in (A, G). Note that in this case the 0-definable relations of $(A/G)_{\text{ind}}$ are also its definable relations. The (abelian) group operation of the quotient group A/G is definable in $(A/G)_{\text{ind}}$. Our aim is to show that this is the full story, despite the ordering of A:

Proposition 5.1.9. *The definable relations of* $(A/G)_{\text{ind}}$ *are exactly the definable relations of the abelian group* A/G.

We shall prove this by working in a more general two-sorted setting. We introduce the 2-sorted language $\mathcal{L}^2$ with sorts s, t and the following nonlogical symbols:

- a function symbol $+$ of arity $(s, s; s)$,
- a function symbol $-$ of arity $(s; s)$,
- a relation symbol $<$ of arity (s, s),
- a constant symbol 0 of arity (s),
- a function symbol $+$ of arity $(t, t; t)$,
- a function symbol $-$ of arity $(t; t)$,
- a constant symbol 0 of arity (t),
- a function symbol P of arity $(s; t)$.

We consider the symbol $+$ of arity $(s, s; s)$ as distinct from the symbol $+$ of arity $(t, t; t)$, and likewise with the symbols $-$ and 0. In this section $(A, B; P)$, sometimes decorated with subscripts or accents, denotes an $\mathcal{L}^2$-structure, with A the underlying set of sort s and B the underlying set of sort t, and P the interpretation of the symbol P; likewise with $(C, D; Q)$.

An $\mathcal{L}^2$-homomorphism $f : (A, B; P) \to (C, D; Q)$ consists of two maps

$$f_s : A \to C \text{ and } f_t : B \to D,$$

but for simplicity we denote both these maps by f.

Let A_0 be an ordered abelian group, B_0 an abelian group, and $P_0 : A_0 \to B_0$ a surjective abelian group morphism such that $(A_0, \ker P_0)$ is a model of $T_{\text{o}}(e, f)$. Let $\mathcal{L}^2(A_0, B_0)$ be the language $\mathcal{L}^2$ augmented by a name of sort s for each element of A_0, a name of sort t for each element of B_0, and for every $n > 0$ a function symbol $\frac{1}{n}$ of arity $(s; s)$ and a relation symbol D_n of arity (t). We construe $(A_0, B_0; P_0)$ as an $\mathcal{L}^2(A_0, B_0)$-structure by interpreting the relation symbol D_n as nB_0 and the function symbol $\frac{1}{n}$ as follows:

$$\frac{1}{n}(a) = \begin{cases} b & \text{if } b \in A_0 \text{ and } a = nb, \\ 0 & \text{if there is no } b \in A_0 \text{ with } a = nb. \end{cases}$$

Hence we have an isomorphism

$$x \mapsto \frac{1}{n}x \ : \ nA_0 \to A_0$$

of ordered abelian groups, which induces a group isomorphism

$$\big(nA_0 \cap \ker P_0\big)/n \ker P_0 \ \to \ \frac{1}{n}\big(nA_0 \cap \ker P_0\big)/\ker P_0.$$

As $\big(nA_0 \cap \ker P_0\big)/n \ker P_0$ is a subgroup of the finite group $\ker P_0/n\ker P_0$, we deduce that the group $\frac{1}{n}\big(nA_0 \cap \ker P_0\big)/\ker P_0$ is finite. Since for all $a_0 \in A_0$ we have $na_0 \in \ker P_0$ if and only if $a_0 \in \frac{1}{n}\big(nA_0 \cap \ker P_0\big)$, we conclude:

Lemma 5.1.10. *Let $n > 0$. There are only finitely many $b \in B_0$ such that $nb = 0$.*

Consider the $\mathcal{L}^2$-structures $(A, B; P)$ where

(1) A is an ordered abelian group containing A_0 as an ordered subgroup;

(2) B is an abelian group containing B_0 as a subgroup;

(3) $P : A \to B$ is a surjective group morphism extending P_0;

(4) $(A, \ker P)$ is a model of $T_{\mathrm{o}}(e, f)$;

(5) for every $n > 0$, $n\frac{1}{n}(a) = a$ if $a \in nA$, and $\frac{1}{n}(a) = 0$ if $a \notin nA$;

(6) for every $n > 0$ and $b \in B$, if $nb = 0$, then $b \in B_0$;

(7) for every $n > 0$ and $b \in B$, $D_n(b)$ if and only if $b \in nB$;

(8) A_0 is pure in A and B_0 is pure in B.

Construing each such $(A, B; P)$ as $\mathcal{L}^2(A_0, B_0)$-structure in the obvious way, we define $T^2(A_0, B_0)$ to be the $\mathcal{L}^2(A_0, B_0)$-theory of these $(A, B; P)$. Note that every model of $T^2(A_0, B_0)$ is isomorphic to a model $(A, B; P)$ as above; all models of $T^2(A_0, B_0)$ referred to below are tacitly assumed to have this form. So if $(A, B; P)$ is a model of $T^2(A_0, B_0)$, then $(A_0, B_0; P_0)$ *is* a substructure of $(A, B; P)$ (not just isomorphic to a substructure of $(A, B; P)$), and

$$[n]A = [n]A_0 \text{ and } [n]\ker P = [n]\ker P_0$$

for every $n > 0$.

Let $(A, B; P)$ and $(C, D; Q)$ be models of $T^2(A_0, B_0)$ such that $(C, D; Q)$ is $|A|^+$-saturated, and let A' be a proper pure subgroup of A containing A_0 with an embedding

$$f : (A', B; P') \to (C, D; Q),$$

where P' is the restriction of P to A'. Note that then

$$[n]A' = [n]A \text{ and } [n] \ker P' = [n] \ker P$$

for every $n > 0$. The following lemma is direct a consequence of Lemma 4.2.1.

Lemma 5.1.11. *Let $a \in \ker P \setminus A'$. Then there is $c \in \ker Q$ such that for every $a' \in A'$, $a'' \in \ker P'$, $m, n > 0$, and $k, l \in \mathbb{Z}$*

(1) $a' + ka \in mA \Leftrightarrow f(a') + kc \in mC$,

(2) $a'' + la \in n \ker P \Leftrightarrow f(a'') + lc \in n \ker Q$.

Next we consider the case when a is not necessarily in $\ker P$.

Lemma 5.1.12. *Let $a \in A \setminus A'$ and $c_1 \in C \setminus f(A')$ be such that $f(P(a)) = Q(c_1)$. Then there is $c \in C$ such that for every $a' \in A'$, $n > 0$, and $k \in \mathbb{Z}$*

$$a' + ka \in nA \Leftrightarrow f(a') + kc \in nC,$$

and $f(P(a)) = Q(c)$.

Proof. We need to find $c_2 \in \ker Q$ such that for every $a' \in A'$, $n > 0$, and $k \in \mathbb{Z}$

$$a' + ka \in nA \Leftrightarrow f(a') + k(c_1 + c_2) \in nC.$$

By the usual arguments, it suffices to find $c_2 \in \ker Q$ such that for every $a' \in A'$ and $n > 0$

$$a \equiv a' \mod nA \Leftrightarrow c_1 + c_2 \equiv f(a') \mod nC.$$

Let $a \equiv a'_i \mod n_iA$, where $a'_i \in A'$ and $n_i > 0$ for $i = 1, \ldots, t$. By saturation, it is enough to find $c_2 \in \ker Q$ such that $c_1 + c_2 \equiv f(a'_i) \mod n_iC$ for $i = 1, \ldots, t$. By Lemma 4.1.3, the existence of such c_2 is equivalent to

$$f(a'_i) \equiv f(a'_j) \mod n_iC + n_jC \text{ and } c_1 \equiv f(a'_i) \mod \ker Q + n_iC$$

for every $i, j \in \{1, \ldots, t\}$. Given $i, j \in \{1, \ldots, t\}$, the first of these conditions is true since

$$a_i \equiv a_j \mod n_iA + n_jA.$$

For the second part note that

$$Q(c_1) - Q(f(a'_i)) = f(P(a - a'_i)) = Q(n_ic_i)$$

for some $c_i \in C$. Thus $c_1 - f(a_i) \in \ker Q + n_iC$. □

Lemma 5.1.13. *Let $a \in A \setminus A'$ and $c_1 \in C \setminus f(A')$ be such that $f(P(a)) = Q(c_1)$ and for every $a' \in A'$, $n > 0$, and $k \in \mathbb{Z}$*

$$a' + ka \in nA \Leftrightarrow f(a') + kc_1 \in nC.$$

Then there is $c \in C$ such that for every $a' \in A'$, $n > 0$, and $k \in \mathbb{Z}$:

1. *$a' + ka \in nA \Leftrightarrow f(a') + kc \in nC$,*
2. *$f\big(P(\frac{a'+ka}{n})\big) = Q\big(\frac{f(a')+kc}{n}\big)$, whenever $a' + ka \in nA$.*

Proof. We need to find $c_2 \in C$ such that for every $a' \in A'$, $n > 0$, and $k \in \mathbb{Z}$

$$a' + ka \in nA \Leftrightarrow f(a) + k(c_1 + c_2) \in nC,$$

and

$$f\big(P(\frac{a' + ka}{n})\big) = Q\big(\frac{f(a') + k(c_1 + c_2)}{n}\big),$$

whenever $a' + ka \in nA$.

Let $n_1, \ldots, n_t > 0$, $k_1, \ldots, k_t \in \mathbb{Z}$, and $a'_1, \ldots, a'_t \in A'$ be such that $\gcd(n_i, k_i) = 1$ and $a'_i + k_ia \in n_iA$ for $i = 1, \ldots, t$. By saturation it is enough to find $c_2 \in C$ such that for $i = 1, \ldots, t$, $k_ic_2 \in n_iC$, and

$$f\big(P(\frac{a'_i + k_ia}{n_i})\big) = Q\big(\frac{f(a'_i) + k_i(c_1 + c_2)}{n_i}\big).$$

Note that for $i = 1, \ldots, t$, $n_i\big(f(P(\frac{a_i'+k_i a}{n_i})) - Q(\frac{f(a_i')+k_i c_1}{n_i})\big) = 0$, thus

$$f(P(\frac{a_i' + k_i a}{n_i})) - Q(\frac{f(a_i') + k_i c_1}{n}) \in B_0.$$

For $i = 1, \ldots, t$ let $a_0^{(i)} \in A_0$ be such that

$$f\big(P(\frac{a_i' + k_i a}{n_i})\big) - Q\big(\frac{f(a_i') + k_i c_1}{n_i}\big) = Q\big(a_0^{(i)}\big).$$

We need to find $c_2 \in C$ such that $k_i c_2 \in n_i a_0^{(i)} + n_i \ker Q$ for $i = 1, \ldots, t$. This happens if and only if $\gamma c_2 \in a_0 + \delta \ker Q$, where $\gamma := \gcd(k_1, \ldots, k_t)$, $\delta := \gcd(n_1, \ldots, n_t)$, and $a_0 \in \ker P_0$ depending on $n_1, \ldots, n_t, k_1, \ldots, k_t, a_0^{(1)}, \ldots a_0^{(t)}$. Then there is a c_2 fitting our purposes if and only if $a_0 \in \gamma C + \delta \ker Q$. This holds since

$$\ker P_0 = \gamma \ker P_0 + \delta \ker P_0 \subseteq \gamma C + \delta \ker P_0.$$

□

Proposition 5.1.14. *$T^2(A_0, B_0)$ admits QE.*

Proof. We use Theorem 2.2.1. So let $(A, B; P)$ and $(C, D; Q)$ be two models of $T^2(A_0, B_0)$ such that $(C, D; Q)$ is $|A|^+$-saturated, and let $(A', B'; P')$ be a proper substructure of $(A, B; P)$ with an embedding $f : (A', B'; P') \to (C, D; Q)$. It suffices to find $(A'', B''; P'') \subseteq (A, B; P)$ containing $(A', B'; P')$, and an embedding of $(A'', B''; P'')$ into $(C, D; Q)$ extending f such that either $A'' \neq A'$ or $B'' \neq B'$. Put $C' := f(A')$ and $D' := f(B')$.

First suppose that $B' \neq B$. If B' is not pure in B, then take $b \in B \setminus B'$ such that $nb = b' \in B'$, where $n > 0$ is the smallest positive natural number such that $nB' \neq nB \cap B'$. Then take $d \in D$ such that $nd = f(b')$. Then it is easy to see that there is a group morphism $B' + \mathbb{Z}b \to D' + \mathbb{Z}d$, extending f by taking b to d. So we have an $\mathcal{L}^2(A_0, B_0)$-embedding $(A', B' + \mathbb{Z}b; P') \to (C, D; Q)$. Hence we may assume that B' is pure in B. It follows that D' is pure in D.

Now let $b \in B \setminus B'$. Then using Corollary 4.1.2 and saturation take $d \in D \setminus D'$ such that for every $b' \in B'$, $n > 0$ and $k \in \mathbb{Z}$

$$b' + kb \in nB \Leftrightarrow f(b') + kd \in nD.$$

Then we can extend f to an embedding $(A', B' + \mathbb{Z}b; P') \to (C, D; Q)$ by taking b to d. Therefore we may assume that $B' = B$. Hence we can apply Lemmas 5.1.11, 5.1.12 and 5.1.13.

We have two cases:

Case 1: $\ker P \nsubseteq A'$. Let $a \in \ker P \setminus A'$. By Lemma 5.1.11, we get $c \in \ker Q$ such that for every $a' \in A'$, $a'' \in \ker P'$, $m, n > 0$, and $k, l \in \mathbb{Z}$:

(i) $a' + ka \in mA \Leftrightarrow f(a') + kc \in mC$,

(ii) $a'' + la \in n \ker P \Leftrightarrow f(a'') + lc \in n \ker Q$.

Using Lemma 5.1.13, we may assume that

$$f(P(\frac{a' + ka}{n})) = Q(\frac{f(a') + kc}{n}),$$

whenever $a' + ka \in nA$ with $a' \in A$, $n > 0$ and $k \in \mathbb{Z}$. Moreover we may choose c in a way that Nc realizes the cut in C' corresponding to the cut of Na in A' for every $N \in \mathbb{N}$. Hence we have an embedding

$$(A'\langle a \rangle, B; P'') \to (C, D; Q),$$

by taking a to c, where P'' is the restriction of P to $A'\langle a \rangle$.

Case 2: $\ker P \subseteq A'$. Let $a \in A \setminus A'$. Then $a \notin A'\langle \ker P \rangle$. Then using Lemmas 5.1.12 and 5.1.13, we can take $c \in C \setminus C'\langle f(\ker P') \rangle$ such that for every $a' \in A'$, $n > 0$, and $k \in \mathbb{Z}$

$$a' + ka \in nA \Leftrightarrow f(a') + kc_1 \in nC$$

and

$$f(P(\frac{a' + ka}{n})) = Q(\frac{f(a') + kc}{n}),$$

whenever $a' + ka \in nA$ with $a' \in A$, $n > 0$ and $k \in \mathbb{Z}$. Once again we may modify c in a way that Nc realizes the cut in C' corresponding to the cut of Na in A' for every $N \in \mathbb{N}$. Hence we have the isomorphism

$$(A'\langle a \rangle, B'; P'') \to (C'\langle c \rangle, D'; Q''),$$

by taking a to c, finishing the proof of the proposition. (Here, as usual, P'' is the restriction of P to $A'\langle a \rangle$, and Q'' is the restriction of Q to $C'\langle c \rangle$).

□

We now prove Proposition 5.1.9 as an easy consequence of Proposition 5.1.14.

Proof. Let (A, G) be a model of $T_{\mathrm{o}}(e, f)$, and $X \subseteq (A/G)^n$ definable in $(A/G)_{\mathrm{ind}}$. Then $(A, A/G; \pi)$ is a model of $T^2(A, A/G)$, and X is definable in $(A, A/G; \pi)$. Then by Proposition 5.1.14, X is quantifier-free definable in $(A, A/G; \pi)$. Hence X is definable in the abelian group A/G. $\square$

5.2 The case that G is regularly discrete

Let G be an ordered abelian group. Then G is said to be *regularly discrete* if G has a smallest positive element and $|G/pG| = p$ for every p. It is easy to see that if G has a least positive element, then the following are equivalent:

(1) G is regularly discrete;

(2) for every p and all $g < h$ in G such that (g, h) has at least p elements we have $pG \cap (g, h) \neq \emptyset$;

(3) G is a $\mathbb{Z}$-group.

For these facts, see [22] and [25].

Let A be regularly dense ordered abelian group and G an ordered subgroup of A with a smallest positive element 1. If

$$\text{for all } a \in A \text{ there is } g \in G \text{ such that } g \leq a < g + 1, \tag{5.2.1}$$

then G is regularly discrete and cofinal in A. Note that (5.2.1) is satisfied if for each $a \in A$ there are $g_1, g_2 \in G$ such that $g_1 \leq a < g_2$, and the interval (g_1, g_2) contains only finitely many elements of G.

In the rest of this section A is a regularly dense ordered abelian group, G is a subgroup of A such that G has a smallest positive element 1 *and (5.2.1) holds for A and G, and $[p]A$ and $[p]G$ are finite for each p.* (These assumptions are clearly satisfied if A is a dense subgroup of $\mathbb{R}^{>0}$ of finite rank and G is a non-trivial cyclic subgroup of $\mathbb{R}^{>0}$.)

Let B be a regularly dense ordered abelian group and H an ordered subgroup of B with a smallest positive element such that (5.2.1) holds with B, H in the place of A, G. Also suppose that (A, G) and (B, H) have the same system of prime invariants and are κ-saturated, where κ is an uncountable cardinal. We denote the smallest positive element of G and H by 1.

Let $\mathcal{L}_{\mathrm{oab1}}$ be the language augmenting $\mathcal{L}_{\mathrm{oab}}$ by a constant symbol 1; similarly, $\mathcal{L}_{\mathrm{oab1}}(V)$ extends $\mathcal{L}_{\mathrm{oab}}(V)$ by a constant symbol 1.

Let $\mathcal{I}$ be the set of isomorphisms

$$\iota : (A', G') \to (B', H')$$

where (A', G') and (B', H') are $\mathcal{L}_{\text{oab1}}(V)$-substructures of (A, G) and (B, H) of cardinality less than κ such that

(1) A', B' are pure subgroups of A, B respectively,

(2) for any $a' \in A'$, there is $g' \in G'$ such that $g' \leq a' < g' + 1$, and for any $b' \in B'$, there is $h' \in H'$ such that $h' \leq b' < h' + 1$,

(3) for each p,

$$[p]A' \geq [p]A, \quad [p]B' \geq [p]B.$$

Remarks. Let (A', G') be a substructure of (A, G) satisfying (1). Then $G' = A' \cap G$ is a pure subgroup of G. Therefore for each p, $[p]G' \leq [p]G = p$, and since G' has a smallest positive element, namely 1, we get $[p]G' = [p]G = p$. Thus G' is a regularly discrete ordered abelian group. If in addition to (1), A', B' also satisfy (3), then for each p

$$[p]A' = [p]A \text{ and } [p]B' = [p]B.$$

We proceed to prove that $\mathcal{I}$ has the back-and-forth property. So let

$$\iota : (A', G') \to (B', H')$$

be in $\mathcal{I}$. Given $a \in A \setminus A'$, we extend ι to an isomorphism in $\mathcal{I}$ which contains a in its domain.

First let $a \in G$. By Lemma 4.2.1, we can find $b \in H$ such that for all $a' \in A'$, $g' \in G'$, $m, n > 0$ and $k, l \in \mathbb{Z}$:

$$a' + ka \in mA \Longleftrightarrow \iota(a') + kb \in mB, \tag{5.2.2}$$

$$g' + la \in nG \Longleftrightarrow \iota(g') + lb \in nH. \tag{5.2.3}$$

Now we modify b such that for each $N \in \mathbb{N}^{>0}$, Nb realizes the cut in B' corresponding to the cut of Na in A': Let $a'_1, a'_2 \in A'$ and $N \in \mathbb{N}^{>0}$ such that $a'_1 < Na < a'_2$. Since $a \notin A'$, using (2), we get

$$a'_1 < a'_1 + 1 < a'_1 + 2 < \cdots < Na < \cdots < a'_2 - 2 < a'_2 - 1 < a'_2.$$

Thus there are infinitely many elements of G' in the interval (a'_1, a'_2). So the interval $\big(\iota(a'_1) - Nb, \iota(a'_2) - Nb\big)$ contains infinitely many elements of H. Therefore as H is regularly discrete, for every $n > 0$,

$$\big(\iota(a'_1) - Nb, \iota(a'_2) - Nb\big) \cap nH$$

is nonempty. Thus by saturation $\big(\iota(a'_1) - Nb, \iota(a'_2) - Nb\big)$ contains an element h of H which is divisible by all $n > 0$ in H. Now $Nb + h \in \big(\iota(a'_1), \iota(a'_2)\big)$, and for all $a' \in A'$, $g' \in G'$, $m, n > 0$ and $k, l \in \mathbb{Z}$:

$$a' + ka \in mA \Longleftrightarrow \iota(a') + k(b + \frac{h}{N}) \in mB,$$

$$g' + la \in nG \Longleftrightarrow \iota(g') + l(b + \frac{h}{N}) \in nH.$$

Hence by saturation, b satisfying (5.2.2) and (5.2.3) can be chosen in a way that for each $N > 0$, Nb realizes the cut in B' corresponding to the cut of Na in A'. Now we can extend ι to an $\mathcal{L}_{\text{oab1}}(V)$-isomorphism

$$(A'\langle a\rangle_A, G'\langle a\rangle_G) \to (B'\langle b\rangle_B, H'\langle b\rangle_H).$$

As noted in the regularly dense case, (1) and (3) hold with $A'\langle a\rangle_A$,$G'\langle a\rangle_G$, $B'\langle b\rangle_B$, and $H'\langle b\rangle_H$ in the place of A', G', B' and H'. It is also easy to check (2) with $A'\langle a\rangle_A, G'\langle a\rangle_G, B'\langle b\rangle_B$, and $H'\langle b\rangle_H$.

If $a \in A'\langle G\rangle_A$, we extend ι to the desired isomorphism exactly as in the case that G is regularly dense.

Now let $a \in A \setminus A'\langle G\rangle_A$. Let $g \in G$ be such that $g \leq a < g + 1$. Using the first case, choose $h \in H$ such that ι extends to an isomorphism

$$\tilde{\iota} : \big(A'\langle g\rangle_A, G'\langle g\rangle_G\big) \to \big(B'\langle h\rangle_B, H'\langle h\rangle_H\big).$$

Now let $b \in B \setminus B'\langle H\rangle_B$ such that for each $N \in \mathbb{N}^{>0}$, Nb realizes the cut in B' corresponding to the cut of Na in A'. So we can extend $\tilde{\iota}$ to an isomorphism

$$\big(A'\langle a, g\rangle_A, G'\langle g\rangle_G\big) \to \big(B'\langle b, h\rangle_B, H'\langle h\rangle_H\big).$$

It is again easy to check that (1) and (3) are satisfied for $A'\langle a, g\rangle_A, G'\langle g\rangle_G$, $B'\langle b, h\rangle_B, H'\langle h\rangle_H$ in the place of A', G', B', H'. So it remains to show (2) with $A'\langle a, g\rangle_A, G'\langle g\rangle_G, B'\langle b, h\rangle_B$ and $H'\langle h\rangle_H$ in the place of A', G', B' and H'. Let $\frac{a'+ka+lg}{m} \in A'\langle a, g\rangle$. Note that we may assume that $k \geq 0$. Let $g' \in G'$ be such that

$g' \leq a' < g'+1$. Hence

$$g' + (k+l)g \leq a' + ka + lg < g' + (k+l)g + (k+1).$$

Then we choose i, j from $\{0, 1, \ldots, m\}$ such that $g' + (k+l)g - i \in mG$, and $g' + (k+l)g + (k+1) + j \in mG$. Hence

$$\frac{g' + (k+l)g - i}{m} \leq \frac{a' + ka + lg}{m} < \frac{g' + (k+l)g + (k+1) + j}{m}.$$

Thus we get (2), as there are finitely many elements of $G'\langle g \rangle$ between $\frac{g'+(k+l)g-i}{m}$ and $\frac{g'+(k+l)g+(k+1)+j}{m}$. This finishes the proof of the fact that $\mathcal{I}$ has the back-and-forth property.

Now we prove a series of results analogous to the results in the previous section.

Corollary 5.2.1. *Let A' be a dense pure subgroup of A containing 1, and let $G' := A' \cap G$. Suppose that for each $a' \in A'$, there is $g' \in G'$ with $g' \leq a' < g'+1$, and that (A, G) and (A', G') have the same system of prime invariants. Then $(A', G') \preceq (A, G)$.*

Proof. Let κ be an uncountable cardinal larger than $|A'|$, and let (A^*, G^*) be a κ-saturated elementary extension of (A', G'). We may assume that (A, G) is also κ-saturated. Then we have a back-and-forth system $\mathcal{I}$ between (A, G) and (A^*, G^*) containing the identity map on (A', G'), which yields the desired result. □

For a family $e = \big(e(p)\big)$ of natural numbers indexed by the primes, let T_e be the $\mathcal{L}_{\text{oab1}}(V)$-theory whose models are of the form (A, G) where A is a regularly dense ordered abelian group, G is a subgroup of A with the smallest positive element 1 such that for any $a \in A$, there is $g \in G$ with $g \leq a < g+1$, and the system of prime invariants of (A, G) is $e, \mathbf{1}$. By an earlier remark if (A, G) is a model of T_e, then G is automatically regularly discrete.

To construct a model of T_e, first assume that e is not identically 0. Let $A = \bigoplus_p \mathbb{Z}_{(p)}^{e(p)}$. Embed A into the additive group of $\mathbb{R}$, and equip it with the order induced from $\mathbb{R}$. This way A becomes a regularly dense ordered abelian group with $[p]A = p^{e(p)}$ for every p. Take a positive element, a of A, and let $G = \mathbb{Z}a$. Then (A, G) becomes a model of T_e.

If $e(p) = 0$ for each p, then take $A = \mathbb{Q}$ with the order induced from $\mathbb{R}$, and $G = \mathbb{Z}$. Then (A, G) is a model of T_e.

Thus T_e is consistent for each $e = \big(e(p)\big)$.

Let $x = (x_1, \dots, x_m)$ be a tuple of variables. Recall that in the previous subsection a special formula in x was defined to be an $\mathcal{L}_{\text{oab}}(V)$-formula $\exists y\big(V(y) \wedge \theta_V(y) \wedge \psi(x, y)\big)$, where $y = (y_1, \dots, y_n)$, and $\theta(y)$ and $\psi(x, y)$ are $\mathcal{L}_{\text{oab}}$-formulas. In this subsection a *special formula in* x is an $\mathcal{L}_{\text{oab1}}(V)$ formula $\exists y\big(V(y) \wedge \theta_V(y) \wedge \psi(x, y)\big)$, where $\theta(y)$ is an $\mathcal{L}_{\text{oab1}}$-formula, and $\psi(x, y)$ is an $\mathcal{L}_{\text{oab}}$-formula.

Lemma 5.2.2. *Each $\mathcal{L}_{\text{oab1}}(V)$-formula is $T_{\mathbf{0}}$-equivalent to a boolean combination of special formulas.*

Proof. Let (A, G) and (B, H) be two $\aleph_1$-saturated models of $T_{\mathbf{0}}$; thus taking $\kappa = \aleph_1$ there is a back-and-forth system $\mathcal{I}$ between (A, G) and (B, H). Let $a \in A^m$ and $b \in B^m$ satisfy the same special formulas. It suffices to prove that a and b satisfy the same types in (A, G) and (B, H) respectively. To show this, it is enough to find an element $\iota : (A', G') \to (B', H')$ of $\mathcal{I}$ with $a \in (A')^m$, $b \in (B')^m$ and $\iota(a) = b$.

Let $\text{rk}(G\langle a \rangle_A / G) = r$. We may assume that $\text{rk}(G\langle a_1, \dots, a_r \rangle_A / G) = r$, and $a_i \in G\langle a_1, \dots, a_r \rangle_A$ for $r < i \le m$. Then since a and b satisfy the same special formulas, we have

$$\text{rk}(H\langle b \rangle_B / H) = r \text{ and } b_i \in H\langle b_1, \dots, b_r \rangle_B \text{ for } r < i \le m.$$

Take $g'_1, \dots, g'_r \in G$ such that $g'_i \le a_i < g'_i + 1$ for all $i = 1, \dots, r$. Now let G' be a countable pure subgroup of G containing $g'_1, \dots, g'_r$ and 1 such that $\text{rk}(G'\langle a \rangle_A / G') = r$. Note that $G'\langle a \rangle_A$ is divisible, being a pure subgroup of the divisible group A. Thus $[p]G'\langle a \rangle_A = [p]A = 1$ for every p. Hence $(G'\langle a \rangle_A, G')$ is a countable substructure of (A, G) satisfying (1), (2) and (3)(with $G'\langle a \rangle_A$ in the place of A').

Enumerate G' as $g = (g_0, g_1, \dots)$, and let $y = (y_0, y_1, \dots)$ be a countable tuple of distinct variables. If $\theta_1(y), \dots, \theta_k(y)$ are $\mathcal{L}_{\text{oab1}}$-formulas and $\phi_1(x, y), \dots,\ \phi_k(x, y)$ are $\mathcal{L}_{\text{oab}}$-formulas such that $G \models \theta_i(g)$ and $A \models \phi_i(a, g)$ for $i = 1, \dots, k$, then

$$(A, G) \models \exists y\big(V(y) \wedge \theta_V(y) \wedge \phi(a, y)\big),$$

where $\theta(y) := \bigwedge_{i=1}^{k} \theta_i(y)$, and $\phi(x, y) := \bigwedge_{i=1}^{k} \phi_i(x, y)$. Therefore

$$(B, H) \models \exists y\big(V(y) \wedge \theta_V(y) \wedge \phi(b, y)\big).$$

So we have a partial y-type over b in (B, H), consisting of formulas $\theta_V(y)$ and $\phi(b, y)$ where $G \models \theta(g)$ and $A \models \phi(a, g)$. Thus by saturation there is a countable tuple $h = (h_0, h_1, \dots)$ realizing this type. Now let H' be $\{h_0, h_1, \dots\}$. Then H' is a countable pure subgroup of H containing 1 which is $\mathcal{L}_{\text{oab1}}$-isomorphic to G' such that $\text{rk}(H'\langle b \rangle_B / H') = r$, $[p]H' = [p]H$ for every p, and $H'\langle b \rangle_B$ is divisible. Hence $(H'\langle b \rangle_B, H')$ is a countable substructure of (B, H) satisfying (1), (2), and (3)(with $H'\langle b \rangle_B$ in the place of B'). Moreover there is an ordered group isomorphism $G'\langle a\rangle_A \to H'\langle b\rangle_B$ sending g_i to h_i for $i = 0, 1 \ldots$, and a to b. This gives an $\mathcal{L}_{\text{oab1}}(V)$-isomorphism

$$\iota : (G'\langle a\rangle_A, G') \to (H'\langle b\rangle_B, H'),$$

which is in $\mathcal{I}$. □

We prove the next corollary and the proposition after it by using the following well-known fact.

Fact. The theory of $\mathbb{Z}$-groups admits quantifier elimination in the language $\mathcal{L}_{\text{oab1}}$ augmented by a new unary predicate E_n for each $n > 0$, interpreted in each $\mathbb{Z}$-group G as the subgroup nG. In particular the theory of $\mathbb{Z}$-groups is complete.

The following consequence of Proposition 5.2.2 is already proven in [18].

Corollary 5.2.3. *The $\mathcal{L}_{\text{oab}}(V)$-theory $T_{\mathbf{0}}$ is complete.*

Proof. Let σ be an $\mathcal{L}_{\text{oab}}(V)$-sentence. Then by Lemma 5.2.2, we may assume that σ is of the form

$$\exists y\big(V(y) \wedge \theta_V(y) \wedge \psi(y)\big),$$

where $\theta(y)$ is an $\mathcal{L}_{\text{oab1}}$-formula and $\psi(y)$ is an $\mathcal{L}_{\text{oab}}$-formula. Moreover we may choose $\psi(y)$ to be quantifier-free since the theory of divisible ordered abelian groups admits quantifier elimination. Thus σ holds in a model (A, G) of $T_{\mathbf{0}}$ if and only if $\exists y(\theta(y) \wedge \psi(y))$ holds in G. As the theory of $\mathbb{Z}$-groups is complete, the completeness of $T_{\mathbf{0}}$ follows. □

Proposition 5.2.4. *Let (A, G) be a model of $T_{\mathbf{0}}$. Then every subset of A^m definable in (A, G) is a boolean combination of subsets of A^m defined in (A, G) by formulas $\exists y\big(V(y) \wedge \psi(x, y)\big)$, where $\psi(x, y)$ is a quantifier-free formula in the language $\mathcal{L}_{\text{oab}}$ augmented by names for the elements of A.*

Proof. By Lemma 5.2.2, it is enough to prove that the subsets of A^m defined by special formulas are of the required form. Let $\theta(y)$ be an $\mathcal{L}_{\text{oab1}}$-formula and $\phi(x, y)$ an $\mathcal{L}_{\text{oab}}$-formula. Our task is to find a quantifier-free formula $\psi(x, z)$ in the language $\mathcal{L}_{\text{oab}}$ augmented by names for elements of A, with z a tuple of variables such that $\big(\exists y\big(V(y) \wedge \theta_V(y) \wedge \phi(x, y)\big)$ is equivalent in (A, G) to $\exists z\big(V(z) \wedge \psi(x, z)\big)$.

By the fact above, the set $\{g \in G^n : G \models \theta(g)\}$ is a boolean combination of subsets of G^n of the form

$$\{g \in G^n : k_1 g_1 + \cdots + k_n g_n = N\},$$

$$\{g \in G^n : k_1 g_1 + \cdots + k_n g_n < N\},$$

$$\{g \in G^n : k_1g_1 + \cdots + k_ng_n \in dG\},$$

where $k_1, \ldots, k_n \in \mathbb{Z}$, $N \in \mathbb{N}$ and d is a positive integer.

Since G is regularly discrete $|G/dG|$ is an integer, and hence the complement of the set

$$\{g \in G^n : k_1g_1 + \cdots + k_ng_n \in dG\}$$

in G is a finite union of sets of the form

$$\{g \in G^n : k_1g_1 + \cdots + k_ng_n \in g_0 + dG\},$$

where $g_0 \in G$.

Thus we get the desired $\psi(x, z)$, which can be chosen to be quantifier-free as the theory of divisible ordered abelian groups admits quantifier elimination. □

Now we add a unary predicate D_n for each $n > 0$ to $\mathcal{L}_{\text{oab1}}(V)$ to get the language $\mathcal{L}^*_{\text{oab1}}(V)$. Extend T_e to an $\mathcal{L}^*_{\text{oab1}}(V)$-theory T^*_e by adding the defining axiom (5.1.1) for each $n > 0$.

Corollary 5.2.5. *The $\mathcal{L}^*_{\text{oab1}}(V)$-theory T^*_e is model complete.*

Proof. Let $(A', G') \subseteq (A, G)$ be two models of T^*_e. Then by 5.1.4, we get that $A' \preceq A$. In particular A' is pure in A. Now by Corollary 5.2.1, $(A', G') \preceq (A, G)$. □

Next for each p, add $p^{e(p)}$ many distinct constant symbols, and a unary function symbol λ to $\mathcal{L}^*_{\text{oab1}}(V)$ to get the language $\mathcal{L}^{**}_{\text{oab1}}(V)$. Extend T^*_e to an $\mathcal{L}^{**}_{\text{oab1}}(V)$-theory, T^{**}_e by adding defining axioms (5.1.3) for the new constant symbols, and the following axiom for λ:

$$\lambda(x) = y \leftrightarrow \big(V(y) \wedge (y \leq x < y + 1)\big).$$

Note that if (A, G) is a model of T^*_e, then (A, G) has a unique expansion (A, G, λ) to a model of T^{**}_e, and that in this expansion the function $\lambda : A \to A$ has image G.

Proposition 5.2.6. *The $\mathcal{L}^{**}_{\text{oab1}}(V)$-theory T^{**}_e admits quantifier elimination.*

Proof. We apply Theorem 2.2.1. So let (A, G), and (B, H) be two models of T^{**}_e, such that (A, G) is countable and (B, H) is $\aleph_1$-saturated, also let (A', G') be a $\mathcal{L}^{**}_{\text{oab1}}(V)$-substructure of (A, G) with an embedding $\iota : (A', G') \to (B, H)$. Let $B' := \iota(A')$ and $H' := \iota(G')$. We need to extend ι properly to an embedding of an

$\mathcal{L}^{**}_{\text{oab1}}(V)$-substructure of (A, G). If A' is pure in A, then we can extend ι properly using the back-and-forth system constructed earlier in this section. If G' is not pure in G, then take the smallest positive natural number n such that $nG' \neq nG \cap G'$, and let $g' \in nG \setminus nG'$, say $g' = ng$. Then $\iota(g') \in nH$, say $\iota(g') = nh$. Then h realizes the cut in B' corresponding to the cut in A' realized by g. Also we have an abelian group isomorphism $A' + \mathbb{Z}g \to B' + \mathbb{Z}h$, which gives an isomorphism

$$(A' + \mathbb{Z}g, G + \mathbb{Z}g) \to (B' + \mathbb{Z}h, H' + \mathbb{Z}h),$$

between substructures of (A, G) and (B, H).

So we may assume that G' is pure in G. Now take $a' \in A'$ such that $a' \in nA \setminus A'$. Then $\iota(a') \in nB \setminus B'$, say $\iota(a') = nb$. Note that b realizes the cut in B' corresponding to the cut of a in A'. Thus there is an ordered group isomorphism $A'\langle a\rangle_A \to B'\langle b\rangle_B$. Note that by the purity of G' in G we get that $\lambda(a) \in G'$. Hence we get an $\mathcal{L}^{**}_{\text{oab1}}(V)$-isomorphism

$$(A'\langle a\rangle, G'') \to (B'\langle b\rangle, H''),$$

where $G'' := A'\langle a\rangle \cap G$ and $H'' := B'\langle b\rangle \cap H$. $\square$

Chapter 6

The real field with two multiplicative groups

In this chapter we use the notations of [9]. In particular, $\mathcal{L}_{\mathrm{o}}$ is the language $\{0, 1, -, +, \cdot, <\}$ of ordered rings with the sublanguage $\mathcal{L}_{\mathrm{om}} = \{1, \cdot, <\}$ of ordered monoids. Let $\mathcal{L}_{\mathrm{o}}(U, V)$ be the language $\mathcal{L}_{\mathrm{o}}$ augmented by two distinct unary relation symbols U, V.

Fix a real closed field R, and two subgroups Δ, Γ of $R^{>0}$ with the Mann property such that $\Gamma \subseteq \Delta$, and $[p]\Delta$, and $[p]\Gamma$ are finite for each p. It follows that $[n]\Delta$ and $[n]\Gamma$ are finite for all $n > 0$; for $n > 0$ we put $k_n := [n]\Delta$ and $l_n := [n]\Gamma$, and we fix sets $\{\delta_{n1}, \ldots, \delta_{nk_n}\}$ and $\{\gamma_{n1}, \ldots, \gamma_{nl_n}\}$ of coset representatives for $\Delta^{[n]}$ in Δ and $\Gamma^{[n]}$ in Γ respectively.

Now let $\mathcal{L}_{\mathrm{o}}(\Delta, U, V)$ be the language obtained by adding a name for each element of Δ to $\mathcal{L}_{\mathrm{o}}(U, V)$. We define the *Mann axioms* of Δ and Γ as in the Introduction. Here we repeat the *ordering axioms* from [9] for Δ: given any tuple $k = (k_1, \ldots, k_n)$ of integers, and any tuple $\delta = (\delta_1, \ldots, \delta_n)$ of elements of Δ, $n \geq 1$, the ordering axiom for k, δ is the sentence

$$k_1\delta_1 + \cdots + k_n\delta_n > 0$$

if this sentence holds in R, and otherwise it is the sentence

$$k_1\delta_1 + \cdots + k_n\delta_n \leq 0.$$

We first prove two lemmas.

Lemma 6.0.7. *Let K be a field with subgroups A, G of $K^{>0}$ such that $G \subseteq A$, and let K' be a subfield of K with subgroups A', G' of $(K')^{>0}$ such that $A' \subseteq A$ and $G' = A' \cap G$. Suppose that for all $q_1, \ldots, q_n \in \mathbb{Q}^\times$ the equation $q_1x_1 + \cdots + q_nx_n = 1$ has the same nondegenerate solutions in A' as in A, and that K' and $\mathbb{Q}(A)$ are free over $\mathbb{Q}(A')$. Then K' and $\mathbb{Q}(G)$ are free over $\mathbb{Q}(G')$.*

Proof. Let $g_1, \ldots, g_m \in G$ be algebraically dependent over K'. Since K' and $\mathbb{Q}(A)$ are free over $\mathbb{Q}(A')$, $g_1, \ldots, g_m$ are algebraically dependent over $\mathbb{Q}(A')$. So there are $n \geq 1$, nonzero integers $k_1, \ldots, k_n$, elements

$a_1, \ldots, a_n \in A'$, and distinct multi-indices $i_1, \ldots, i_n$ of length m such that $k_1 a_1 g^{i_1} + \cdots + k_n a_n g^{i_n} = 0$, where $g = (g_1, \ldots, g_m)$. We also assume that $n \geq 1$ is minimal with this property. As a result of this,

$$(\frac{a_1 g^{i_1}}{a_n g^{i_n}}, \ldots, \frac{a_{n-1} g^{i_{n-1}}}{a_n g^{i_n}})$$

is a nondegenerate solution of $(-\frac{k_1}{k_n})x_1 + \cdots + (-\frac{k_{n-1}}{k_n})x_{n-1} = 1$ in A. Thus we get $\frac{a_1 g^{i_1}}{a_n g^{i_n}} \in A'$. Therefore $g^{i_1 - i_n}$ is in A', hence in G'. So $g_1, \ldots, g_k$ are algebraically dependent over $\mathbb{Q}(G')$, which proves that K' and $\mathbb{Q}(G)$ are free over $\mathbb{Q}(G')$. $\square$

From this proof we can conclude the following.

Corollary 6.0.8. *Let K, A, G, K', A', G' be as in the previous lemma and let $g_1, \ldots, g_m \in G$ be algebraically dependent over K'. Then $g_1, \ldots, g_m$ are multiplicatively dependent over G'.*

By Lemmas 3.1.3, and 3.1.4 from the Introduction, we get the following:

Lemma 6.0.9. *Let K, A, G, K', A', G' be as in the previous lemma. Suppose also that A' is pure in A. Then (K', A', G') is a substructure of (K, A, G).*

Remark. As noted in [9], in the setting of the lemma above it follows that $\mathbb{Q}(A)$ is a regular extension of $\mathbb{Q}(A')$, and that $\mathbb{Q}(G)$ is a regular extension of $\mathbb{Q}(G')$. Hence by Theorem 2.1.1, K' and $\mathbb{Q}(A)$ are linearly disjoint over $\mathbb{Q}(A')$, and K' and $\mathbb{Q}(G)$ are linearly disjoint over $\mathbb{Q}(G')$.

6.1 The case that G is dense

Let $\mathrm{RCF}_1(\Delta, \Gamma)$ be the $\mathcal{L}_o(\Delta, U, V)$-theory whose models are of the form $\big(K, A, G, (\delta')_{\delta \in \Delta}\big)$ such that:

(1) K is a real closed field, G and A are dense subgroups of $K^{>0}$ such that $G \subseteq A$, and A/G is infinite,

(2) $\delta \mapsto \delta' : \Delta \to A$ and $\gamma \mapsto \gamma' : \Gamma \to G$ are group homomorphisms, and if $\delta' \in G$, then $\delta \in \Gamma$,

(3) for every $n > 0$ and $a \in A$, a is congruent to one of $\delta'_{n1}, \ldots, \delta'_{nk_n}$ modulo $A^{[n]}$,

(4) for every $n > 0$ and $g \in G$, g is congruent to one of $\gamma'_{n1}, \ldots, \gamma'_{nl_n}$ modulo $G^{[n]}$,

(5) $\big(K, (\delta')_{\delta \in \Delta}\big)$ satisfies the ordering axioms for Δ,

(6) $\big(K, A, (\delta')_{\delta \in \Delta}\big)$ satisfies the Mann axioms for Δ,

(7) $\big(K, G, (\gamma')_{\gamma \in \Gamma}\big)$ satisfies the Mann axioms for Γ.

If $\big(K, A, G, (\delta')_{\delta\in\Delta}\big)$ is a model of $\mathrm{RCF}_1(\Delta, \Gamma)$, then Δ and Γ are embedded into A and G respectively as ordered groups. From now on, we identify the images of Δ and Γ in K by themselves, and write a model of $\mathrm{RCF}_1(\Delta, \Gamma)$ as $\big(K, A, G, (\delta)_{\delta\in\Delta}\big)$ or $\big(K, A, G, (\delta)\big)$.

A dense multiplicative subgroup of $K^{>0}$ is regularly dense when K is a real closed field. Combining this fact with (1), (3) and (4), the results of Section 5.1 can be applied to (A, G) whenever (K, A, G) is a model of $\mathrm{RCF}_1(\Delta, \Gamma)$.

If Δ and Γ are dense in $R^{>0}$, and Δ/Γ is infinite, then the structure $\big(R, \Delta, \Gamma, (\delta)_{\delta\in\Delta}\big)$ is a model of $\mathrm{RCF}_1(\Delta, \Gamma)$. We classify the models of $\mathrm{RCF}_1(\Delta, \Gamma)$ up to elementary equivalence in the next theorem.

Theorem 6.1.1. *Let $\big(K, A, G, (\delta)_{\delta\in\Delta}\big)$ and $\big(L, B, H, (\delta)_{\delta\in\Delta}\big)$ be two models of $\mathrm{RCF}_1(\Delta, \Gamma)$. Then they are elementarily equivalent if and only (A, G) and (B, H) have the same system of prime invariants, and for all $\delta \in \Delta$, $\gamma \in \Gamma$, and $m, n > 0$:*

$$\delta \textit{ is an } m^{\text{th}} \textit{ power in } A \Longleftrightarrow \delta \textit{ is an } m^{\text{th}} \textit{ power in } B,$$

$$\gamma \textit{ is an } n^{\text{th}} \textit{ power in } G \Longleftrightarrow \gamma \textit{ is an } n^{\text{th}} \textit{ power in } H.$$

Proof. The forward direction is clear, so we only prove the converse direction. For this purpose, assume that $[p]A = [p]B$, $[p]G = [p]H$ for every p, and for all $\delta \in \Delta$, $\gamma \in \Gamma$, and $m, n > 0$:

$$\delta \text{ is an } m^{\text{th}} \text{ power in } A \Longleftrightarrow \delta \text{ is an } m^{\text{th}} \text{ power in } B,$$

$$\gamma \text{ is an } n^{\text{th}} \text{ power in } G \Longleftrightarrow \gamma \text{ is an } n^{\text{th}} \text{ power in } H.$$

We show that $\big(K, A, G, (\delta)\big)$ and $\big(L, B, H, (\delta)\big)$ are elementarily equivalent by constructing a nonempty back-and-forth system between them. We may assume that $\big(K, A, G, (\delta)\big)$ and $\big(L, B, H, (\delta)\big)$ are κ-saturated for some infinite cardinal $\kappa > |\Delta|$.

Define $\mathrm{Sub}(K, A, G)$ to be the set of all $\mathcal{L}_{\mathrm{o}}(U, V)$-structures (K', A', G') where K' is a real closed subfield of K of cardinality less than κ, A', G' are subgroups of A, G containing Δ, Γ respectively such that A' is pure in A, $G' = A' \cap G$, and K' and $\mathbb{Q}(A)$ are free over $\mathbb{Q}(A')$.

Suppose that $(K', A', G') \in \mathrm{Sub}(K, A, G)$. Then by Lemmas 6.0.7 and 6.0.9, we know that K' and $\mathbb{Q}(G)$ are free over $\mathbb{Q}(G')$, and that (K', A', G') is a substructure of (K, A, G). Also it follows from Axioms (3) and (4) that for each p, $[p]A' \geq [p]A$ and $[p]G' \geq [p]G$, and hence $[p]A' = [p]A$ and $[p]G' = [p]G$ by purity.

Define $\mathrm{Sub}(L,B,H)$ in a similar way, and let $\mathcal{I}$ be the set of all isomorphisms

$$\iota : (K', A', G') \to (L', B', H'),$$

between elements of $\mathrm{Sub}(K,A,G)$ and $\mathrm{Sub}(L,B,H)$ fixing elements of Δ pointwise. We show that $\mathcal{I}$ is a nonempty back-and-forth system.

We first show that $\mathcal{I} \neq \emptyset$: Let

$$\begin{aligned}
A' &:= \{a \in A : a^n \in \Delta \text{ for some } n > 0\}, \quad K' := \mathbb{Q}(\Delta)^{\mathrm{rc}} \subseteq K,\\
B' &:= \{b \in B : b^n \in \Delta \text{ for some } n > 0\}, \quad L' := \mathbb{Q}(\Delta)^{\mathrm{rc}} \subseteq L,\\
G' &:= \{g \in G : g^n \in \Gamma \text{ for some } n > 0\},\\
H' &:= \{h \in H : h^n \in \Gamma \text{ for some } n > 0\}.
\end{aligned}$$

Then by the second part of axiom (2), $A' \cap G = G'$ and $B' \cap H = H'$, and thus $(K', A', G') \in \mathrm{Sub}(K,A,G)$ and $(L', B', H') \in \mathrm{Sub}(L,B,H)$, and the the ordered field isomorphism $K' \cong L'$ fixing Δ pointwise is in $\mathcal{I}$.

Now we proceed to prove that $\mathcal{I}$ is a back-and-forth system. Let

$$\iota : (K', A', G') \to (L', B', H')$$

be in $\mathcal{I}$ and $\alpha \in K \setminus K'$. We need to find an element of $\mathcal{I}$ with α in its domain. We distinguish five cases.

Case 1: $\alpha \in G$. Using Section 5.1, we can find $\beta \in H$ such that for every natural number N, $N\beta$ realizes the cut in B' corresponding to the cut of $N\alpha$ in A' and for all $a' \in A'$, $g' \in G'$, $m, n > 0$ and $k, l \in \mathbb{Z}$:

$$a'\alpha^k \text{ is an } m^{\text{th}} \text{ power in } A \Longleftrightarrow \iota(a')\beta^k \text{ is an } m^{\text{th}} \text{ power in } B.$$

$$g'\alpha^l \text{ is an } n^{\text{th}} \text{ power in } G \Longleftrightarrow \iota(g')\beta^l \text{ is an } n^{\text{th}} \text{ power in } H.$$

Since H is regularly dense in $L^{>0}$ we can choose β to realize the cut in L' corresponding to the cut of α in K'. Now we can extend ι to

$$\big(K'(\alpha)^{\mathrm{rc}}, A'\langle\alpha\rangle_A, G'\langle\alpha\rangle_G\big) \to \big(L'(\beta)^{\mathrm{rc}}, B'\langle\beta\rangle_B, H'\langle\beta\rangle_H\big), \quad \alpha \mapsto \beta.$$

Note that $K'(\alpha)^{\mathrm{rc}}$ and $\mathbb{Q}(A)$ are free over $\mathbb{Q}(A'\langle\alpha\rangle_A)$ by Lemma 3.1.3, and similarly.$L'(\beta)^{\mathrm{rc}}$ and $\mathbb{Q}(B)$ are free over $\mathbb{Q}(B'\langle\beta\rangle_B)$. Now it is easy to check that the isomorphism is in $\mathcal{I}$.

Case 2: $\alpha \in A'\langle G\rangle$. Take $g \in G$ such that $\alpha \in A'\langle g\rangle_A$. Then by the previous step there is $h \in H$, and an isomorphism $\big(K'(g)^{\mathrm{rc}}, A'\langle g\rangle_A, G'\langle g\rangle_G\big) \to \big(L'(h)^{\mathrm{rc}}, B'\langle h\rangle_B, H'\langle h\rangle_H\big)$ in $\mathcal{I}$. This finishes this case since $\alpha \in K'(g)^{\mathrm{rc}}$.

Case 3: $\alpha \in A \setminus A'\langle G\rangle$. Then there is $\beta \in B \setminus B'\langle H\rangle$ with an ordered abelian group isomorphism $A'\langle\alpha\rangle \to B'\langle\beta\rangle$. As in Case 1, using regular density, we may choose β to realize the cut in L' corresponding to the cut of α in K'. So we have an isomorphism

$$\big(K'(\alpha)^{\mathrm{rc}}, A'\langle\alpha\rangle_A, G'\big) \to \big(L'(\beta)^{\mathrm{rc}}, B'\langle\beta\rangle_B, H'\big),$$

which happens to be in $\mathcal{I}$ again by Lemma 3.1.3.

Case 4: $\alpha \in K'(A)^{\mathrm{rc}}$. Then $\alpha \in K'(a_1,\dots,a_k)^{\mathrm{rc}}$ for some $a_1,\dots,a_k \in A$. Therefore we can apply some of the previous steps k times in succession to get the desired isomorphism.

Case 5: $\alpha \notin K'(A)^{\mathrm{rc}}$. By saturation and using that B is small, $L \setminus L'(B)^{\mathrm{rc}}$ is dense in L. Thus we may choose $\beta \notin L'(B)^{\mathrm{rc}}$ realizing the cut in L' corresponding to the cut of α in K'. In this case we have an isomorphism $(K'(\alpha)^{\mathrm{rc}}, A', G') \to (L'(\beta)^{\mathrm{rc}}, B', H')$ which is again in $\mathcal{I}$. This finishes the last case and the proof of the theorem. □

Remark. As a result of this theorem we get that $\mathrm{RCF}_1(\Delta,\Gamma)$ is complete if and only if both Δ and Γ are divisible.

The proof above has the following consequence:

Corollary 6.1.2. *Let $(K', A', G') \subseteq (K, A, G)$ be $\mathcal{L}_o(U,V)$-structures where K' is a real closed field, A', G' are dense subgroups of $(K')^{>0}$ with the Mann property such that A'/G' is infinite, and the prime invariants $[p]A'$ and $[p]G'$ are finite for every p. Suppose that $\big(K, A, G, (a')_{a'\in A'}\big)$ is a model of $\mathrm{RCF}_1(A', G')$, A' is a pure subgroup of A, and K' and $\mathbb{Q}(A)$ are free over $\mathbb{Q}(A')$. Then $(K', A', G') \preceq (K, A, G)$.*

Proof. Let (K^*, A^*, G^*) be a κ-saturated elementary extension of (K', A', G') where κ is an uncountable cardinal greater than $|K'|$. We may also assume that (K, A, G) is κ-saturated. By the proof of 6.1.1, there is a back-and-forth system $\mathcal{I}$ between substructures of (K^*, A^*, G^*), and (K, A, G) containing the identity map on (K', A', G'). Thus

$$(K^*, A^*, G^*) \equiv_{K'} (K, A, G),$$

and hence $(K', A', G') \preceq (K, A, G)$. □

6.2 The case that G is regularly discrete

In this section we assume that Δ is a dense subgroup of $\mathbb{R}^{>0}$ and Γ has a smallest element greater than 1, say $\mathrm{b}(\Gamma)$. This is for the ease of presentation. It follows from these assumptions that $\Gamma = \mathrm{b}(\Gamma)^{\mathbb{Z}}$, and that for every $\alpha \in \mathbb{R}^{>0}$ there is $\gamma \in \Gamma$ such that

$$\gamma \leq \alpha < \gamma\, \mathrm{b}(\Gamma).$$

Let $\mathrm{RCF}_2(\Delta, \Gamma)$ be the $\mathcal{L}_{\mathrm{o}}(\Delta, U, V)$-theory whose models are of the form $\big(K, A, G, (\delta')_{\delta\in\Delta}\big)$ such that:

(1) K is a real closed field, A and G are subgroups of $K^{>0}$ such that $G \subseteq A$, A is dense in $K^{>0}$, and $\mathrm{b}(\Gamma)'$ is the smallest element of G greater than 1,

(2) $\delta \mapsto \delta' : \Delta \to A$ and $\gamma \mapsto \gamma' : \Gamma \to G$ are group homomorphisms, and if $\delta' \in G$, then $\delta \in \Gamma$,

(3) for every $a \in A$, there is $g \in G$ such that $g \leq a < g\, \mathrm{b}(\Gamma)'$,

(4) for every $n > 0$ and $a \in A$, a is equivalent to one of $\delta'_{n1}, \ldots, \delta'_{nk_n}$ modulo $A^{[n]}$,

(5) $\big(K, (\delta')_{\delta\in\Delta}\big)$ satisfies the ordering axioms for Δ,

(6) $\big(K, A, (\delta')_{\delta\in\Delta}\big)$ satisfies the Mann axioms for Δ,

(7) $\big(K, G, (\gamma')_{\gamma\in\Gamma}\big)$ satisfies the Mann axioms for Γ.

As for $\mathrm{RCF}_1(\Delta, \Gamma)$, every model of $\mathrm{RCF}_2(\Delta, \Gamma)$ contains copies of Δ and Γ; we identify these copies with Δ and Γ and write a model of $\mathrm{RCF}_2(\Delta, \Gamma)$ as $\big(K, A, G, (\delta)_{\delta\in\Delta}\big)$ or $\big(K, A, G, (\delta)\big)$. In contrast to the previous section, $(\mathbb{R}, \Delta, \Gamma)$ is a model of $\mathrm{RCF}_2(\Delta, \Gamma)$.

Let $\big(K, A, G, (\delta)\big)$ be a model of $\mathrm{RCF}_2(\Delta, \Gamma)$. Then A is regularly dense and G is regularly discrete as ordered abelian groups; so Section 5.2 applies to the pair (A, G). It is easy to check that Γ is a pure subgroup of G.

Also note that for every $\alpha \in K^{>0}$, there is $g \in G$ such that $g \leq \alpha < g\, \mathrm{b}(\Gamma)$ since A is dense in $K^{>0}$. Moreover there is a unique such g for given $\alpha \in K^{>0}$. Hence we can define a function $\lambda : K \to G$ as follows:

$$\lambda(\alpha) := \begin{cases} 0 & \text{if } \alpha \leq 0, \\ g & \text{if } \alpha > 0 \text{ and } g \leq \alpha < g\, \mathrm{b}(\Gamma). \end{cases}$$

A non-zero element $\alpha \in K$ is contained in G if and only if $\lambda(\alpha) = \alpha$. Let D be a domain contained in K such that $\lambda(D^{>0}) = D^{>0} \cap G$, and let K' be the fraction field of D. Then it is easy to see that

$$\lambda((K')^{>0}) = (K')^{>0} \cap G \text{ and } \lambda\big((K'')^{>0}\big) = \langle \lambda((K')^{>0}) \rangle_G, \quad K'' := (K')^{\mathrm{rc}}.$$

We say that a subfield K' of K is *closed under* λ if $\lambda((K')^{>0}) = (K')^{>0} \cap G$. Note that $\mathbb{Q}(\Delta) \subseteq K$ is closed under λ.

Theorem 6.2.1. *Let* $\big(K, A, G, (\delta)_{\delta \in \Delta}\big)$ *and* $\big(L, B, H, (\delta)_{\delta \in \Delta}\big)$ *be two models of* $\mathrm{RCF}_2(\Delta, \Gamma)$*. Then they are elementarily equivalent if and only if for every* p*,* $[p]A = [p]B$*, and for all* $\delta \in \Delta$ *,* $\gamma \in \Gamma$*, and* $m, n > 0$*:*

$$\delta \textit{ is an } m^{\text{th}} \textit{ power in } A \iff \delta \textit{ is an } m^{\text{th}} \textit{ power in } B.$$

Proof. It is clear that the forward direction holds. So let $\big(K, A, G, (\delta)\big)$ and $\big(L, B, H, (\delta)\big)$ be such that for every p, $[p]A = [p]B$, and for all $\delta \in \Delta$, $\gamma \in \Gamma$, and $m, n > 0$:

$$\delta \text{ is an } m^{\text{th}} \text{ power in } A \iff \delta \text{ is an } m^{\text{th}} \text{ power in } B.$$

We also assume that they are κ-saturated for some infinite cardinal κ greater than $|\Delta|$. In order to show that they are elementarily equivalent, we proceed in analogy with the proof of Theorem 6.1.1. So define $\mathrm{Sub}(K, A, G)$ to be the set of all $\mathcal{L}_{\mathrm{o}}(U, V)$-structures (K', A', G') such that K' is a real closed subfield of K of cardinality less than κ, A' is a pure subgroup of A containing Δ, $G' = A' \cap G$, K' and $\mathbb{Q}(A)$ are free over $\mathbb{Q}(A')$, and for all $\alpha' \in (K')^{>0}$, there is $g' \in G'$ such that

$$g' \leq \alpha' < g' \, \mathrm{b}(\Gamma).$$

By Lemmas 6.0.7 and 6.0.9, these assumptions yield that K' and $\mathbb{Q}(G)$ are free over $\mathbb{Q}(G')$, and that (K', A', G') is a substructure of (K, A, G). Thus the last condition is equivalent to K' being closed under λ. Also by Axiom (4), we get that for every p, $[p]A' \geq [p]A$, and hence $[p]A' = [p]A$ since A' is pure in A.

Define $\mathrm{Sub}(L, B, H)$ similarly, and let $\mathcal{I}$ be the set of all isomorphisms

$$\iota : (K', A', G') \to (L', B', H')$$

between elements of $\mathrm{Sub}(K, A, G)$ and $\mathrm{Sub}(L, B, H)$ fixing elements of Δ pointwise. We show that $\mathcal{I}$ is a nonempty back-and-forth system. Let

$$A' := \{a \in A : a^n \in \Delta \text{ for some } n > 0\}, \quad K' := \mathbb{Q}(\Delta)^{\mathrm{rc}} \subseteq K$$
$$B' := \{b \in B : b^n \in \Delta \text{ for some } n > 0\}, \quad L' := \mathbb{Q}(\Delta)^{\mathrm{rc}} \subseteq L.$$

Combined with the remarks just before the theorem, the fact that Γ is pure in G gives that $\Gamma = A' \cap G$ and that K' is closed under λ. The other conditions for (K', A', Γ) to be in $\mathrm{Sub}(K, A, G)$ are easy to check. Similarly $(L', B', \Gamma) \in \mathrm{Sub}(L, B, H)$. Now the the ordered field isomorphism $K' \cong L'$ that is identity on Δ gives an isomorphism

$$(K', A', G') \to (L', B', H'),$$

which is in $\mathcal{I}$.

To show that $\mathcal{I}$ is a back-and-forth system, let

$$\iota : (K', A', G') \to (L', B', H')$$

be in $\mathcal{I}$, and $\alpha \in K \setminus K'$.

First let $\alpha \in G$. Using Section 5.2, there is $\beta \in H$ with an isomorphism

$$\big(A'\langle\alpha\rangle, G'\langle\alpha\rangle\big) \to \big(B'\langle\beta\rangle, H'\langle\beta\rangle\big).$$

Now we modify β so that it realizes the cut in L' corresponding to the cut of α in K'. Let $\alpha_1', \alpha_2' \in K'$ such that $\alpha_1' < \alpha < \alpha_2'$. Then

$$\alpha_1' < \alpha_1' \,\mathrm{b}(\Gamma) < \alpha_1' \,\mathrm{b}(\Gamma)^2 < \cdots < \alpha < \cdots < \alpha_2' \,\mathrm{b}(\Gamma)^{-2} < \alpha_2' \,\mathrm{b}(\Gamma)^{-1} < \alpha_2',$$

because otherwise α would be in G'. Then the interval (α_1', α_2') contains infinitely many elements of G. Hence the interval $\big(\iota(\alpha_1')\beta^{-1}, \iota(\alpha_2)\beta^{-1}\big)$ contains infinitely many elements of H. By the regular discreteness of H and saturation we get $h \in \cap_{n>0} H^{[n]}$ from that interval. Therefore $\beta h \in \big(\iota(\alpha_1'), \iota(\alpha_2')\big)$. Thus by saturation we can find the desired modification of β.

Then there is an isomorphism

$$\big(K'(\alpha)^{\mathrm{rc}}, A'\langle\alpha\rangle, G'\langle\alpha\rangle\big) \to \big(L'(\beta)^{\mathrm{rc}}, B'\langle\beta\rangle, H'\langle\beta\rangle\big).$$

Note that $\lambda((K'[\alpha])^{>0}) = G'\alpha^{\mathbb{Z}}$, hence $\lambda((K'(\alpha)^{\mathrm{rc}})^{>0}) = G'\langle\alpha\rangle_G$ by the remarks before the proof, i.e. $K'(\alpha)^{\mathrm{rc}}$ is closed under λ. It is easy to check the other conditions for this isomorphism to be in $\mathcal{I}$ using Lemma 3.1.3.

If $\alpha \in A'\langle G\rangle$, then we get the required isomorphism, by applying the previous step several times.

Let $\alpha \in A \setminus A'\langle G\rangle$. By using the first step, we may assume that $\lambda(a) \in G'$. We, however, need to add more elements of G to G' in order to close $K'(\alpha)^{\mathrm{rc}}$ under λ: for instance it is not necessarily true that $\lambda(a - \lambda(a)) \in G'$.

Let $G^{(i+1)} := \lambda((K'[G^{(i)}, \alpha])^{>0})$ for $i = 1, 2, \dots$, where $G^{(1)} := G'$. Now the cardinality of $G^{\infty} := \bigcup_{i=1}^{\infty} G^{(i)}$ is less than κ, and using the first step we get an extension of ι

$$\big(K'(G^{\infty})^{\mathrm{rc}}, A'\langle G^{\infty}\rangle_A, \langle G^{\infty}\rangle_G\big) \to \big(L'(H^{\infty})^{\mathrm{rc}}, B'\langle H^{\infty}\rangle_A, \langle H^{\infty}\rangle_H\big),$$

where H^{∞} is a subgroup of H. We can find $\beta \in B$ such that we have an $\mathcal{L}_{\mathrm{oab}}(V)$-isomorphism

$$\big(A'\langle G^{\infty}, \alpha\rangle_A, \langle G^{\infty}\rangle_G\big) \to \big(B'\langle H^{\infty}, \beta\rangle_B, \langle H^{\infty}\rangle_H\big),$$

taking α to β. Also we may choose β in a way that we have an isomorphism

$$(K'(G^{\infty}, \alpha)^{\mathrm{rc}}, A'\langle G^{\infty}, \alpha\rangle_A, \langle G^{\infty}\rangle_G) \to (L'(H^{\infty}, \beta)^{\mathrm{rc}}, B'\langle H^{\infty}, \beta\rangle_A, \langle H^{\infty}\rangle_H).$$

This last isomorphism extends ι and is in $\mathcal{I}$.

The cases that $\alpha \in K'(A)^{\mathrm{rc}}$ and $\alpha \in K \setminus K'(A)^{\mathrm{rc}}$ are handled in similar fashions.

□

As in the previous section, we have the following consequence by the back-and-forth system constructed in the proof of 6.2.1:

Corollary 6.2.2. *Suppose that* $\big(K, A, G, (\delta)_{\delta\in\Delta}\big)$ *is a model of* $\mathrm{RCF}_2(\Delta, \Gamma)$*, and that* Δ *is a pure subgroup of* A *and* $[p]\Delta = [p]A$ *for every* p*. Let* $K' \subseteq K$ *be a real closed field containing* Δ *such that* Δ *is dense in* $(K')^{>0}$*, and* K' *and* $\mathbb{Q}(A)$ *are free over* $\mathbb{Q}(\Delta)$*. Then* $(K', \Delta, \Gamma) \preceq (K, A, G)$*.*

Corollary 6.2.3. *The set* $3^{\mathbb{Z}}$ *is not definable in* $(\mathbb{R}, 2^{\mathbb{Z}}3^{\mathbb{Z}}, 2^{\mathbb{Z}})$.

Proof. Take a real closed field $K \supset \mathbb{R}$ and $\alpha > 0$ from $K \setminus \mathbb{R}$. Consider the $\mathcal{L}_{\mathrm{o}}(V)$-structures $(\mathbb{R}, 2^{\mathbb{Z}}3^{\mathbb{Z}}, 2^{\mathbb{Z}})$ and $(K, 2^{\mathbb{Z}}3^{\mathbb{Z}}\alpha^{\mathbb{Q}}, 2^{\mathbb{Z}})$. These structures satisfy the conditions of the corollary above with $K' = \mathbb{R}$, $\Delta = 2^{\mathbb{Z}}3^{\mathbb{Z}}$, $A = 2^{\mathbb{Z}}3^{\mathbb{Z}}\alpha^{\mathbb{Q}}$, and $\Gamma = G = 2^{\mathbb{Z}}$. Therefore

$$(\mathbb{R}, 2^{\mathbb{Z}}3^{\mathbb{Z}}, 2^{\mathbb{Z}}) \preceq (K, 2^{\mathbb{Z}}3^{\mathbb{Z}}\alpha^{\mathbb{Q}}, 2^{\mathbb{Z}}).$$

Now we claim that $3^{\mathbb{Z}}$ is not definable in $(\mathbb{R}, 2^{\mathbb{Z}}3^{\mathbb{Z}}, 2^{\mathbb{Z}})$. For a contradiction assume that it is definable in $(\mathbb{R}, 2^{\mathbb{Z}}3^{\mathbb{Z}}, 2^{\mathbb{Z}})$, say by the formula $\varphi(x)$ in the language $\mathcal{L}_{\mathrm{o}}(U, V)$ augmented by a name for each element of $\mathbb{R}$. Then

$$(\mathbb{R}, 2^{\mathbb{Z}}3^{\mathbb{Z}}, 2^{\mathbb{Z}}) \models \forall x\big((V(x) \wedge \varphi(x)) \rightarrow x = 1\big),$$

$$(\mathbb{R}, 2^{\mathbb{Z}}3^{\mathbb{Z}}, 2^{\mathbb{Z}}) \models \forall x\big(U(x) \rightarrow \exists y \exists z (V(y) \wedge \varphi(z) \wedge x = yz)\big)$$

and

$$(\mathbb{R}, 2^{\mathbb{Z}}3^{\mathbb{Z}}, 2^{\mathbb{Z}}) \models \exists x\big(0 < x \wedge \varphi(x) \wedge \forall y((0 < y \wedge \varphi(y)) \rightarrow x \leq y)\big).$$

Since $2^{\mathbb{Z}}$ does not have a discrete complement in $2^{\mathbb{Z}}3^{\mathbb{Z}}\alpha^{\mathbb{Q}}$, we get a contradiction. Hence $3^{\mathbb{Z}}$ is not definable in $(\mathbb{R}, 2^{\mathbb{Z}}3^{\mathbb{Z}}, 2^{\mathbb{Z}})$. □

This corollary illustrates that the structure $(\mathbb{R}, 2^{\mathbb{Z}}, 3^{\mathbb{Z}})$ cannot be handled using the results from this section.

Chapter 7

Mann pairs

Throughout this chapter we let Ω be an ambient algebraically closed field, taken to be $\mathbb{C}$ if suggested by the context. Also, Γ is a subgroup of the multiplicative group $\Omega^\times$, and $\mathbf{k}$ a subfield of Ω. For $r \in \Omega$ and $\vec{s} = (s_1, \dots, s_n) \in \Omega^n$ we put $r\vec{s} := (rs_1, \dots, rs_n) \in \Omega^n$. In particular, if $\alpha \in \Gamma$ and $\vec{\gamma} \in \Gamma^n$, then $\alpha\vec{\gamma} \in \Gamma^n$. Let $a_1, \dots, a_n \in \Omega$, $n \geq 1$, and consider the equation

$$a_1x_1 + \cdots + a_nx_n = 1. \tag{$*$}$$

A *solution* of $(*)$ is a tuple $(s_1, \dots, s_n) \in \Omega^n$ such that $a_1s_1 + \cdots + a_ns_n = 1$; such a solution is said to be *nondegenerate* if $\sum_{i \in I} a_is_i \neq 0$ for all nonempty $I \subseteq \{1, \dots, n\}$, and is said to be in Γ if $(s_1, \dots, s_n) \in \Gamma^n$.

Definition 7.0.4. Call $(\mathbf{k}, \Gamma)$ a *Mann pair* if for each $n \geq 1$ there is a finite $E \subseteq \Gamma^n$ such that for all $a_1, \dots, a_n \in \mathbf{k}^\times$ all non-degenerate solutions of $(*)$ in Γ lie in E.

Let $\mathbb{U}$ be the multiplicative group of complex roots of unity, and let $A \subseteq \mathbb{C}$ be the field of complex algebraic numbers. Theorem 1 of [17] implies that $(\mathbb{Q}, \mathbb{U})$ is a Mann pair; conversely, Corollary 7.4.1 below says that if $(\mathbb{Q}, \Gamma)$ is a Mann pair and $\Gamma \subseteq A^\times$, then $\Gamma \subseteq \mathbb{U}$. Thus $(\mathbb{Q}, (1+\sqrt{2})^{\mathbb{Z}})$ is not a Mann pair, although $(1+\sqrt{2})^{\mathbb{Z}} \subseteq A^\times$ has the Mann property and $\mathbb{Q}^\times \cap (1+\sqrt{2})^{\mathbb{Z}} = \{1\}$. In Section 7.1 we indicate simple examples of Mann pairs, such as:

(1) $(A, \exp(A))$ is a Mann pair (Lindemann's theorem).

(2) If $v : K^\times \to v(K^\times)$ is a valuation on a subfield K of Ω and $\mathbf{k}$ is a subfield of K and Γ a subgroup of $K^\times$ such that v is trivial on $\mathbf{k}$ and injective on Γ, then $(\mathbf{k}, \Gamma)$ is a Mann pair.

A more substantial source of Mann pairs is provided by Theorem 1.0.3 from the Introduction, which we prove in Sections 7.2 and 7.5. Then the chapter continues with more model-theoretic issues. This leads to Theorem 1.0.4 from the Introduction (proved in Section 7.6), as well as a determination of the elementary theory of $(\Omega, \mathbf{k}, \Gamma)$ (in Section 7.7) and a description of the definable relations of $(\Omega, \mathbf{k}, \Gamma)$ (in Section 7.8). In these results we assume that $(\mathbf{k}, \Gamma)$ is a Mann pair and, in some cases, also that $\mathbf{k}$ is algebraically closed.

7.1 Some examples of Mann pairs

It will be useful to consider also *homogeneous* linear equations. Let $n \geq 1$ and let $a_1, \ldots, a_n \in \Omega$. A *solution* of the homogeneous linear equation

$$a_1x_1 + \cdots + a_nx_n = 0$$

is a tuple $(s_1, \ldots, s_n) \in \Omega^n$ such that $a_1s_1 + \cdots + a_ns_n = 0$; such a solution is said to be *non-degenerate* if $\sum_{i \in I} a_is_i \neq 0$ for all nonempty proper subsets I of $\{1, \ldots, n\}$, and is said to be in Γ if $(s_1, \ldots, s_n) \in \Gamma^n$.

7.1.1 Alternative definition of Mann pairs

The equivalence of (1) and (2) in the next lemma expresses being a Mann pair in terms of homogeneous linear equations. Assuming $\mathbf{k}^\times \cap \Gamma$ is finite, the equivalence between (1) and (3) expresses $(\mathbf{k}, \Gamma)$ being a Mann pair purely in terms of the subgroup $G = \mathbf{k}^\times\Gamma$ of $\Omega^\times$ generated by $\mathbf{k}^\times$ and Γ.

Lemma 7.1.1. *Let $G = \mathbf{k}^\times\Gamma$. Then the following are equivalent:*

(1) *$(\mathbf{k}, \Gamma)$ is a Mann pair;*

(2) *for each $n \geq 1$ there is a finite $\Gamma(n) \subseteq \Gamma^n$ such that each non-degenerate solution in Γ of each equation $a_1x_1 + \cdots + a_nx_n = 0$ with $a_1, \ldots, a_n \in \mathbf{k}^\times$ lies in $\alpha\Gamma(n)$ for some $\alpha \in \Gamma$;*

(3) *$\mathbf{k}^\times \cap \Gamma$ is finite, and for each $n \geq 1$ there is a finite $G(n) \subseteq G^n$ such that each non-degenerate solution in G of the equation*

$$x_1 + \cdots + x_n = 1$$

equals $(c_1g_1, \ldots, c_ng_n)$ with $c_1, \ldots, c_n \in \mathbf{k}^\times$ and $(g_1, \ldots, g_n) \in G(n)$.

Proof. Let $n \geq 1$. First note that if $a_1, \ldots, a_n \in \mathbf{k}^\times$ and $(s_1, \ldots, s_n) \in \Omega^n$, then $(s_1, \ldots, s_n)$ is a non-degenerate solution of $a_1x_1 + \cdots + a_nx_n = 1$ iff $(s_1, \ldots, s_n, 1)$ is a non-degenerate solution of

$$a_1x_1 + \cdots + a_nx_n - x_{n+1} = 0.$$

Therefore, if (1) holds, witnessed by E for our given n as in Definition 7.0.4, then (2) holds with

$$\Gamma(n+1) := \{(\gamma_1, \ldots, \gamma_n, 1) : (\gamma_1, \ldots, \gamma_n) \in E\}.$$

Conversely, if (2) holds, take $\Gamma(n+1)$ such that $\gamma_{n+1} = 1$ for all tuples $(\gamma_1, \ldots, \gamma_n, \gamma_{n+1}) \in \Gamma(n+1)$, and then (1) holds with

$$E := \{(\gamma_1, \ldots, \gamma_n) : \ (\gamma_1, \ldots, \gamma_n, 1) \in \Gamma(n+1)\}.$$

Next, observe that if (1) holds, witnessed by E for our given n, then (3) holds with $G(n) := E$. Conversely, if (3) holds, then in (3) we can take $G(n) \subseteq \Gamma^n$, and then (1) holds with

$$E := \{(\alpha_1\gamma_1, \ldots, \alpha_n\gamma_n) \in \Gamma^n : \alpha_1, \ldots, \alpha_n \in \mathbf{k}^\times \cap \Gamma, (\gamma_1, \ldots, \gamma_n) \in G(n)\}. \qquad \square$$

Corollary 7.1.2. *Suppose* $(\mathbf{k}, \Gamma)$ *is a Mann pair, and* Γ' *is a subgroup of* $\mathbf{k}^\times\Gamma$ *such that* $\mathbf{k}^\times \cap \Gamma'$ *is finite. Then* $(\mathbf{k}, \Gamma')$ *is a Mann pair.*

7.1.2 Easy Mann pairs

These are provided by the next lemma.

Lemma 7.1.3. *Suppose* Γ *is torsion-free. Then the following are equivalent:*

(1) *for all* $n \geq 1$ *and* $a_1, \ldots, a_n \in \mathbf{k}^\times$*, the equation* $a_1x_1 + \cdots + a_nx_n = 1$ *has no non-degenerate solution in* Γ *that is different from* $(1, \ldots, 1)$*;*

(2) *whenever* $n \geq 1$ *and* $\gamma_1, \ldots, \gamma_n \in \Gamma$ *are multiplicatively independent, then they are algebraically independent over* $\mathbf{k}$*.*

If these conditions are satisfied, then $(\mathbf{k}, \Gamma)$ *is a Mann pair.*

Proof. Apply Lemma 8.2 in [9] (but $\Gamma,\ E,\ G$ in that lemma are $\{1\},\ \mathbf{k},\ \Gamma$ in the present lemma). $\square$

Since $\pi \notin A$, the group $\exp(A) \subseteq \mathbb{C}^\times$ is torsion-free, and by Lindemann's theorem on exponentials ([15], Appendix 1), condition (2) of the lemma is satisfied with $\mathbf{k} = A \subseteq \mathbb{C}$ and $\Gamma = \exp(A)$. Thus $(A, \exp(A))$ is a Mann pair. Here is another application of the lemma, which applies for example to Hahn fields $K = \mathbf{k}((\Gamma))$.

Corollary 7.1.4. *Let* $v : K^\times \to v(K^\times)$ *be a valuation on a subfield* K *of* Ω*. Suppose* $\mathbf{k}$ *is a subfield of* K *and* Γ *a subgroup of* $K^\times$ *such that* v *is trivial on* $\mathbf{k}$ *and injective on* Γ*. Then condition* (2) *of Lemma 7.1.3 is satisfied, and so* $(\mathbf{k}, \Gamma)$ *is a Mann pair.*

A group with the Mann property can produce a Mann pair in an elementary extension as follows:

Corollary 7.1.5. *Suppose* G *is a subgroup of* $\mathbf{k}^\times$ *with the Mann property,* $(\mathbf{k}^*, G^*)$ *is an elementary extension of* $(\mathbf{k}, G)$*, and* Γ *is a subgroup of* G^* *such that* $\mathbf{k}^\times \cap \Gamma = \{1\}$*. Then* $(\mathbf{k}, \Gamma)$ *is a Mann pair.*

Proof. Note that Γ is torsion-free. Let $a_1,\dots,a_n \in \mathbf{k}^\times$, $n \geq 1$. Then the equation $a_1x_1+\cdots+a_nx_n = 1$ has the same finite number of non-degenerate solutions in G as in G^*, so cannot have non-degenerate solutions in Γ different from $(1,\dots,1)$, the unique element of $G^n \cap \Gamma^n$. Hence condition (1) of Lemma 7.1.3 is satisfied, and so $(\mathbf{k},\Gamma)$ is a Mann pair. ☐

If $\mathbf{k}$ has positive characteristic, then Lemma 7.1.3 is the only source of Mann pairs $(\mathbf{k},\Gamma)$ with torsion-free Γ:

Corollary 7.1.6. *Suppose* $\mathbf{k}$ *has positive characteristic,* Γ *is torsion-free, and* $(\mathbf{k},\Gamma)$ *is a Mann pair. Then conditions* (1) *and* (2) *of Lemma 7.1.3 are satisfied.*

Proof. Let $a_1,\dots,a_n \in \mathbf{k}^\times$, $n \geq 1$, and suppose $(\gamma_1,\dots,\gamma_n) \in \Gamma^n$ is a non-degenerate solution of

$$a_1x_1+\cdots+a_nx_n = 1.$$

Let ϕ be the Frobenius map on $\mathbf{k}$. Then for each m we have a non-degenerate solution

$$\big(\phi^m(\gamma_1),\dots,\phi^m(\gamma_n)\big) \in \Gamma^n$$

of the equation $\phi^m(a_1)x_1+\cdots+\phi^m(a_n)x_n = 1$. By the equivalence (1) $\Leftrightarrow$ (2) of Lemma 7.1.1 this yields that all γ_i are roots of unity, so equal to 1 by torsion-freeness of Γ. ☐

7.2 Mann pairs in function fields of one variable

Let $\mathbf{k}$ be algebraically closed, and let $F \subseteq \Omega$ be a function field of one variable over $\mathbf{k}$, that is, F is a field extension of finite degree of $\mathbf{k}(t)$ for some $t \in \Omega \setminus \mathbf{k}$; in particular, F has transcendence degree 1 over $\mathbf{k}$. Below we use some standard facts about such function fields; for proofs of these facts, see Chapter I, §2 of [14].

Let $\mathcal{R}(F|\mathbf{k})$, the *Riemann space* of F over $\mathbf{k}$, be the set of all valuations $v : F^\times \to \mathbb{Z}$ on F with value group $v(F^\times) = \mathbb{Z}$ that are trivial on $\mathbf{k}$. We let v range over $\mathcal{R}(F|\mathbf{k})$. For each $f \in F^\times$ we have $v(f) \neq 0$ for only finitely many v, and $\sum_v v(f) = 0$. Let $\mathcal{D}(F|\mathbf{k})$ be the group of *divisors* of F over $\mathbf{k}$, that is,

$$\mathcal{D}(F|\mathbf{k}) \ := \ \bigoplus_v \mathbb{Z}v$$

is the free abelian group on the Riemann space of F over $\mathbf{k}$. To $f \in F^\times$ we assign its *principal divisor*

$(f) := \sum_v v(f)v \in \mathcal{D}(F|\mathbf{k})$. The group morphism

$$F^\times \to \mathcal{D}(F|\mathbf{k}), \quad f \mapsto (f)$$

has kernel $\mathbf{k}^\times$. In particular, if Γ is a subgroup of $F^\times$ with $\mathbf{k}^\times \cap \Gamma = \{1\}$, then this morphism is injective on Γ, and so the image of Γ under this morphism is an isomorphic copy of Γ inside the free abelian group $\mathcal{D}(F|\mathbf{k})$. It follows that each such Γ is free as an abelian group.

Given finite $S \subseteq \mathcal{R}(F|\mathbf{k})$, an *$S$-unit* is an element $u \in F^\times$ such that $v(u) = 0$ for all $v \notin S$.

Lemma 7.2.1. *Let L be a finite-dimensional $\mathbf{k}$-linear subspace of F and let $\Gamma \subseteq F^\times$ be finitely generated with $\mathbf{k}^\times \cap \Gamma = \{1\}$. Then $L \cap \Gamma$ is finite.*

Proof. Let $b_1, \ldots, b_m$ be a basis of the $\mathbf{k}$-linear space L, and let $\gamma_1, \ldots, \gamma_n$ generate the group Γ. Take a finite $S \subseteq \mathcal{R}(F|\mathbf{k})$ such that all b_i and γ_j are S-units. Take a natural number d such that $v(b_i) \geq -d$ for all $v \in S$ and $i = 1, \ldots, m$. Then $v(f) \geq 0$ for all $f \in L$ and v outside S, and $v(f) \geq -d$ for all $f \in L$ and $v \in S$.

Suppose now that $\gamma \in L \cap \Gamma$. Then $v(\gamma) = 0$ for v outside S and $v(\gamma) \geq -d$ for $v \in S$. In view of $\sum_v v(\gamma) = 0$, this gives $v(\gamma) \leq |S|d$ for all $v \in S$. It follows that the image of $L \cap \Gamma$ in $\mathcal{D}(F|\mathbf{k})$ is finite, and thus $L \cap \Gamma$ is finite. ☐

Let $u_1, \ldots, u_n \in F$ not be all zero. We define their *height* by

$$H(u_1, \ldots, u_n) := -\sum_v \min\{v(u_1), \ldots, v(u_n)\}.$$

This height is projective: $H(fu_1, \ldots, fu_n) = H(u_1, \ldots, u_n)$ for $f \in F^\times$.

Example. Let $F = \mathbf{k}(t)$ with t transcendental over $\mathbf{k}$. Suppose that the polynomials $u_1, \ldots, u_n \in \mathbf{k}[t]$ have no common zero in $\mathbf{k}$, $n \geq 1$. It is easy to check that then $H(u_1, \ldots, u_n) = \max\{\deg_t u_1, \ldots, \deg_t u_n\}$.

The following important bound is from [6]:

Let Ω have characteristic zero, let g be the genus of the function field $F|\mathbf{k}$, and let S be a finite subset of $\mathcal{R}(F|\mathbf{k})$ and $n \geq 2$. Suppose $u_1, \ldots, u_n$ are S-units and $(u_1, \ldots, u_n)$ is a non-degenerate solution of $x_1 + \cdots + x_n = 0$. Then

$$H(u_1, \ldots, u_n) \leq \frac{1}{2}(n-1)(n-2)\{|S| + \max(0, 2\,\mathrm{g} - 2)\}.$$

In combination with the previous lemma this has the following consequence:

Corollary 7.2.2. *Suppose Ω has characteristic zero and $\Gamma \subseteq F^\times$ is finitely generated with $\mathbf{k}^\times \cap \Gamma = \{1\}$. Then $(\mathbf{k}, \Gamma)$ is a Mann pair.*

Proof. Take a finite $S \subseteq \mathcal{R}(F|\mathbf{k})$ such that all $\gamma \in \Gamma$ are S-units. Let $n \geq 2$ and let $a_1, \dots, a_n \in \mathbf{k}^\times$ and $\gamma_1, \dots, \gamma_n \in \Gamma$ be such that $(\gamma_1, \dots, \gamma_n)$ is a non-degenerate solution of $a_1 x_1 + \cdots + a_n x_n = 0$. Dividing by γ_n we arrange $\gamma_n = 1$, and we need to show that this leaves only finitely many possibilities for $(\gamma_1, \dots, \gamma_{n-1})$. Now by the bound above we have $v(\gamma_i) \geq -N$ for all $v \in S$ and $i = 1, \dots, n-1$, where

$$N := \frac{1}{2}(n-1)(n-2)\{|S| + \max(0, 2\,\mathrm{g} - 2)\},$$

so each γ_i lies in the $\mathbf{k}$-linear subspace

$$L := \{f \in F : \; v(f) \geq 0 \text{ for all } v \notin S, \, v(f) \geq -N \text{ for all } v \in S\}$$

of F. Now L is finite-dimensional by [14], p. 7. In view of Lemma 7.2.1 this gives the desired finiteness. $\square$

Example. To illustrate the effective nature of this proof, assume that Ω has characteristic zero, and consider the case $F = \mathbf{k}(t)$ of a *rational* function field (so $\mathrm{g} = 0$), where Γ is generated as a group by

$$t - c_1, \dots, t - c_M, \qquad \text{with distinct } c_1, \dots, c_M \in \mathbf{k}.$$

Let $n \geq 2$ be given. Call a tuple $(\gamma_1, \dots, \gamma_n) \in \Gamma^n$ *reduced* if all γ_i lie in $\mathbf{k}[t]$ and $\gamma_1, \dots, \gamma_n$ have no common zero in $\mathbf{k}$ when viewed as polynomials in t over $\mathbf{k}$. Let $a_1, \dots, a_n \in \mathbf{k}^\times$. Any non-degenerate solution in Γ of the equation $a_1 x_1 + \cdots + a_n x_n = 0$ can be multiplied by an element of Γ to give a non-degenerate reduced solution $(\gamma_1, \dots, \gamma_n)$. In this situation the inequality from [6] yields

$$\max\{\deg_t \gamma_1, \dots, \deg_t \gamma_n\} \leq \frac{1}{2}(n-1)(n-2)(M+1),$$

which is satisfied by only finitely many reduced tuples $(\gamma_1, \dots, \gamma_n)$, which we can list explicitly. Given a reduced tuple $(\gamma_1, \dots, \gamma_n)$ satisfying the inequality, the existence of $a_1, \dots, a_n \in \mathbf{k}^\times$ such that $(\gamma_1, \dots, \gamma_n)$ is a non-degenerate solution of $a_1 x_1 + \cdots + a_n x_n = 0$ is equivalent (effectively) to the existence of a solution to a certain finite system of linear equations and inequations with coefficients in the field $\mathbb{Q}(c_1, \dots, c_M)$.

7.3 Allowing coefficients from Ω

In this section we consider Γ as acting on itself by multiplication, making Γ into a Γ-set as defined in [9], §4. In particular, any subset of Γ^n definable in the Γ-set Γ is definable in the group Γ. Our aim is to prove the following result and some related facts.

Proposition 7.3.1. *Let $(\mathbf{k}, \Gamma)$ be a Mann pair and $r_1, \ldots, r_n \in \Omega$. Then*

$$\{(x, y) \in \mathbf{k}^n \times \Gamma^n : \ r_1 x_1 y_1 + \cdots + r_n x_n y_n = 0\}$$

is a finite union of sets $X \times Y$ with X a $\mathbf{k}$-linear subspace of $\mathbf{k}^n$ and $Y \subseteq \Gamma^n$ defined in the Γ-set Γ by a conjunction of atoms in the language of Γ-sets.

We first show this for $r_1, \ldots, r_n \in \mathbf{k}$, next for $r_1, \ldots, r_n \in \mathbf{k}(\Gamma)$, and then in general.

7.3.1 Notations

In this section we let $\vec{a} = (a_1, \ldots, a_n)$ and $\vec{b} = (b_1, \ldots, b_n)$ range over $\mathbf{k}^n$, and α, β, γ over Γ, and

$$\vec{\alpha} = (\alpha_1, \ldots, \alpha_n), \quad \vec{\beta} = (\beta_1, \ldots, \beta_n), \quad \vec{\gamma} = (\gamma_1, \ldots, \gamma_n)$$

over Γ^n. Let $n \geq 1$ and put

$$\begin{aligned} \Sigma_n(\mathbf{k}, \Gamma) &:= \{(\vec{a}, \vec{\gamma}) \in \mathbf{k}^n \times \Gamma^n : \ a_1\gamma_1 + \cdots + a_n\gamma_n = 0\}, \\ \Sigma_n(\mathbf{k}, \Gamma; \vec{\gamma}) &:= \{\vec{a} \in \mathbf{k}^n : \ (\vec{a}, \vec{\gamma}) \in \Sigma_n(\mathbf{k}, \Gamma)\}. \end{aligned}$$

Imposing non-degeneracy yields the set $\Sigma_n^{\mathrm{nd}}(\mathbf{k}, \Gamma)$ of all $(\vec{a}, \vec{\gamma}) \in (\mathbf{k}^\times)^n \times \Gamma^n$ such that $\vec{\gamma}$ is a non-degenerate solution of $a_1 x_1 + \cdots + a_n x_n = 0$. We also introduce for each $\vec{\gamma}$ the corresponding section

$$\Sigma_n^{\mathrm{nd}}(\mathbf{k}, \Gamma; \vec{\gamma}) := \{\vec{a} \in (\mathbf{k}^\times)^n : \ (\vec{a}, \vec{\gamma}) \in \Sigma_n^{\mathrm{nd}}(\mathbf{k}, \Gamma)\}.$$

If $(\vec{a}, \vec{\gamma}) \in \Sigma_n^{\mathrm{nd}}(\mathbf{k}, \Gamma)$, then $(\vec{a}, \alpha\vec{\gamma}) \in \Sigma_n^{\mathrm{nd}}(\mathbf{k}, \Gamma)$, so $\Sigma_n^{\mathrm{nd}}(\mathbf{k}, \Gamma)$ is a union of sets of the form $\Sigma_n^{\mathrm{nd}}(\mathbf{k}, \Gamma; \vec{\gamma}) \times \Gamma\vec{\gamma}$. Being a Mann pair can now be expressed as follows:

$(\mathbf{k}, \Gamma)$ *is a Mann pair iff for each $n \geq 1$ there is a finite $\Gamma(n) \subseteq \Gamma^n$ such that*

$$\Sigma_n^{\mathrm{nd}}(\mathbf{k}, \Gamma) = \bigcup_{\vec{\gamma} \in \Gamma(n)} \Sigma_n^{\mathrm{nd}}(\mathbf{k}, \Gamma; \vec{\gamma}) \times \Gamma\vec{\gamma}.$$

Whenever $(\mathbf{k}, \Gamma)$ is a Mann pair we let $\Gamma(n)$ for $n \geq 1$ be as above.

We also want to allow coefficients outside $\mathbf{k}$ and accordingly, given $\vec{r} = (r_1, \dots, r_n) \in \Omega^n$, we let $\Sigma(\vec{r}, \mathbf{k}, \Gamma)$ be the set of all

$$(\vec{a}, \vec{\gamma}) = (a_1, \dots, a_n, \gamma_1, \dots, \gamma_n) \in \mathbf{k}^n \times \Gamma^n$$

such that $\vec{\gamma}$ is a solution of $r_1 a_1 x_1 + \cdots + r_n a_n x_n = 0$, and let $\Sigma^{\mathrm{nd}}(\vec{r}, \mathbf{k}, \Gamma)$ be the set of all $(\vec{a}, \vec{\gamma}) \in (\mathbf{k}^\times)^n \times \Gamma^n$ such that $\vec{\gamma}$ is a non-degenerate solution of $r_1 a_1 x_1 + \cdots + r_n a_n x_n = 0$. In particular, $\Sigma^{\mathrm{nd}}(\vec{r}, \mathbf{k}, \Gamma) = \emptyset$ if $n \geq 2$ and $r_i = 0$ for some $i \in \{1, \dots, n\}$. Also, let $\Sigma^{\mathrm{nd}}(\vec{r}, \mathbf{k}, \Gamma; \vec{\gamma})$ be the set of $\vec{a} \in (\mathbf{k}^\times)^n$ such that $(\vec{a}, \vec{\gamma}) \in \Sigma^{\mathrm{nd}}(\vec{r}, \mathbf{k}, \Gamma)$.

7.3.2 Mann pairs over F

Besides $\mathbf{k}$ we also let F denote a subfield of Ω. Usually, $\mathbf{k}$ serves as basefield, and F will be an extension of $\mathbf{k}$.

We say that $(\mathbf{k}, \Gamma)$ is a *Mann pair over* F if for every tuple $\vec{r} = (r_1, \dots, r_n)$ from $F^\times$, $n \geq 2$, there is a finite subset $\Gamma(\vec{r})$ of Γ^n such that

$$\Sigma^{\mathrm{nd}}(\vec{r}, \mathbf{k}, \Gamma) = \bigcup_{\vec{\gamma} \in \Gamma(\vec{r})} \Sigma^{\mathrm{nd}}(\vec{r}, \mathbf{k}, \Gamma; \vec{\gamma}) \times \Gamma\vec{\gamma}.$$

It is clear that $(\mathbf{k}, \Gamma)$ is a Mann pair iff it is a Mann pair over $\mathbf{k}$. We proceed to show that if $(\mathbf{k}, \Gamma)$ is a Mann pair, then it is a Mann pair over Ω.

Let $\vec{r} = (r_1, \dots, r_n) \in F^n$, $n \geq 1$. Let $\mathcal{P}$ be a partition of $\{1, \dots, n\}$ into distinct sets $I(1), \dots, I(l)$, where $l = l(\mathcal{P}) := |\mathcal{P}|$. Then we define $\Sigma_{\mathcal{P}}(\vec{r}, \mathbf{k}, \Gamma)$ to be the set of all $(a_1, \dots, a_n, \beta_1, \dots, \beta_n) \in \mathbf{k}^n \times \Gamma^n$ such that for $j = 1, \dots, l$ the tuple $(\beta_i)_{i \in I(j)}$ is a non-degenerate solution of $\sum_{i \in I(j)} r_i a_i x_i = 0$. With $\vec{r}(j) := (r_i)_{i \in I(j)}$ for $j = 1, \dots, l$, this means

$$\Sigma_{\mathcal{P}}(\vec{r}, \mathbf{k}, \Gamma) \;=\; \prod_{j=1}^{l} \Sigma^{\mathrm{nd}}(\vec{r}(j), \mathbf{k}, \Gamma).$$

Suppose now that $(\mathbf{k}, \Gamma)$ is an Mann pair over F. Then we have for $j = 1, \dots, l$ a finite $\Gamma(\vec{r}(j)) \subseteq \Gamma^{I(j)}$ such that

$$\Sigma^{\mathrm{nd}}(\vec{r}(j), \mathbf{k}, \Gamma) \;=\; \bigcup_{\vec{\gamma} \in \Gamma(\vec{r}(j))} \Sigma^{\mathrm{nd}}(\vec{r}(j), \mathbf{k}, \Gamma; \vec{\gamma}) \times \Gamma\vec{\gamma}.$$

It follows that

$$\Sigma_{\mathcal{P}}(\vec{r}, \mathbf{k}, \Gamma) \;=\; \prod_{j=1}^{l} \bigcup_{\vec{\gamma} \in \Gamma(\vec{r}(j))} \Sigma^{\mathrm{nd}}(\vec{r}(j), \mathbf{k}, \Gamma; \vec{\gamma}) \times \Gamma\vec{\gamma}.$$

Thus, with $\vec{\gamma}(j) := (\gamma_i)_{i \in I(j)}$ and $\Gamma(\vec{r}) = \Gamma(\vec{r}, \mathcal{P}) := \prod_{j=1}^{l} \Gamma(\vec{r}(j)) \subseteq \Gamma^n$,

$$\begin{aligned}\Sigma_{\mathcal{P}}(\vec{r}, \mathbf{k}, \Gamma) &= \bigcup_{\vec{\gamma} \in \Gamma(\vec{r})} \prod_{j=1}^{l} \left[\Sigma^{\mathrm{nd}}\big(\vec{r}(j), \mathbf{k}, \Gamma; \vec{\gamma}(j)\big) \times \Gamma\vec{\gamma}(j)\right] \\ &\subseteq \bigcup_{\vec{\gamma} \in \Gamma(\vec{r})} \prod_{j=1}^{l} \left[\Sigma\big(\vec{r}(j), \mathbf{k}, \Gamma; \vec{\gamma}(j)\big) \times \Gamma\vec{\gamma}(j)\right].\end{aligned}$$

The last product set is contained in $\Sigma(\vec{r}, \mathbf{k}, \Gamma)$ under an obvious identification, so with $\mathcal{P}$ ranging over the partitions of $\{1, \ldots, n\}$, we get

$$\Sigma(\vec{r}, \mathbf{k}, \Gamma) = \bigcup_{\mathcal{P}} \Sigma_{\mathcal{P}}(\vec{r}, \mathbf{k}, \Gamma) = \bigcup_{\mathcal{P}} \bigcup_{\vec{\gamma} \in \Gamma(\vec{r}, \mathcal{P})} \prod_{j=1}^{l(\mathcal{P})} \left[\Sigma\big(\vec{r}(j), \mathbf{k}, \Gamma; \vec{\gamma}(j)\big) \times \Gamma\vec{\gamma}(j)\right].$$

This yields the following result.

Lemma 7.3.2. *Suppose* $(\mathbf{k}, \Gamma)$ *is a Mann pair over* F*, and* $\vec{r} \in F^n$, $n \geq 1$*. Then* $\Sigma(\vec{r}, \mathbf{k}, \Gamma)$ *is a finite union of sets* $P \times Q$*, where* P *is a* $\mathbf{k}$*-linear subspace of* $\mathbf{k}^n$*, and* $Q \subseteq \Gamma^n$ *is defined in* Γ *by a conjunction of atoms in the language of* Γ*-sets.*

Note that the intersection of two sets of the form $P \times Q$, where P, Q are as in Lemma 7.3.2, is again of the same form. Hence, if $(\mathbf{k}, \Gamma)$ is a Mann pair over F, and $\vec{r}, \vec{s} \in F^n$, $n \geq 1$, then $\Sigma(\vec{r}, \mathbf{k}, \Gamma) \cap \Sigma(\vec{s}, \mathbf{k}, \Gamma)$ is a finite union of sets $P \times Q$, where P, Q are as in Lemma 7.3.2. Next we prove a converse of Lemma 7.3.2.

Lemma 7.3.3. *Let* $\vec{r} \in \Omega^n$*. Suppose that*

$$\Sigma(\vec{r}, \mathbf{k}, \Gamma) = \bigcup_{j=1}^{k} P_j \times Q_j,$$

where $P_1, \ldots, P_k$ *are subsets of* $\mathbf{k}^n$*, and each* Q_j *is a subset of* Γ^n *defined in* Γ *by a conjunction of atoms in the language of* Γ*-sets. Then there is a finite subset* $\Gamma(\vec{r})$ *of* Γ^n *such that*

$$\Sigma^{\mathrm{nd}}(\vec{r}, \mathbf{k}, \Gamma) = \bigcup_{\vec{\gamma} \in \Gamma(\vec{r})} \Sigma^{\mathrm{nd}}(\vec{r}, \mathbf{k}, \Gamma; \vec{\gamma}) \times \Gamma\vec{\gamma}.$$

Proof. We can assume $\Gamma \neq \{1\}$. Let $j \in \{1, \ldots, k\}$ be such that $Q_j \neq \emptyset$. By the discussion in [9] just after Corollary 4.2, we can take $\vec{\gamma} \in \Gamma^n$ and a partition of $\{1, \ldots, n\}$ into distinct sets $I(1), \ldots, I(l)$ such that

$$Q_j = \Gamma\vec{\gamma}(1) \times \cdots \times \Gamma\vec{\gamma}(l) \subseteq \prod_{\lambda=1}^{l} \Gamma^{I(\lambda)} = \Gamma^n,$$

where $\vec{\gamma}(\lambda) := \{(\gamma_i)\}_{i \in I(\lambda)} \in \Gamma^{I(\lambda)}$ for $\lambda = 1, \dots, l$.

Claim. Let $l > 1$, $\vec{a} \in P_j$ and $\vec{\beta} \in Q_j$. Then $(a_1\beta_1, \dots, a_n\beta_n)$ is a degenerate solution of

$$r_1x_1 + \cdots + r_nx_n = 0.$$

Proof of the claim. For each $\alpha \in \Gamma$,

$$\sum_{i \in I(1)} r_ia_i\beta_i + \sum_{i \notin I(1)} r_ia_i\beta_i = 0 = \sum_{i \in I(1)} r_ia_i\beta_i + \sum_{i \notin I(1)} r_ia_i\alpha\beta_i.$$

So $\sum_{i \notin I(1)} r_ia_i\beta_i = \alpha \sum_{i \notin I(1)} r_ia_i\beta_i$ for all $\alpha \in \Gamma$. Hence $\sum_{i \notin I(1)} r_ia_i\beta_i = 0$, and thus $(a_1\beta_1, \dots, a_n\beta_n)$ is a degenerate solution, proving the claim.

As a result of this claim, we get a subset J of $\{1, \dots, k\}$, and a tuple $\vec{\gamma}_j \in \Gamma^n$ for each $j \in J$ such that

$$\Sigma^{\mathrm{nd}}(\vec{r}, \mathbf{k}, \Gamma) \ \subseteq \ \bigcup_{j \in J} P_j \times \Gamma\vec{\gamma}_j.$$

Since obviously $\bigcup_{j \in J} \Sigma^{\mathrm{nd}}(\vec{r}, \mathbf{k}, \Gamma; \vec{\gamma}_j) \times \Gamma\vec{\gamma}_j \subseteq \Sigma^{\mathrm{nd}}(\vec{r}, \mathbf{k}, \Gamma)$, this yields

$$\Sigma^{\mathrm{nd}}(\vec{r}, \mathbf{k}, \Gamma) \ = \ \bigcup_{j \in J} \Sigma^{\mathrm{nd}}(\vec{r}, \mathbf{k}, \Gamma; \vec{\gamma}_j) \times \Gamma\vec{\gamma}_j.$$

□

Note: $P_1, \dots, P_k$ in this lemma are not assumed to be linear subspaces of $\mathbf{k}^n$. Applying the lemma with $\vec{r} = (1, \dots, 1)$, we obtain

Lemma 7.3.4. *Suppose for each $n \geq 1$ the set $\Sigma_n(\mathbf{k}, \Gamma)$ is a finite union of sets $P \times Q$ with $P \subseteq \mathbf{k}^n$ and Q a subset of Γ^n defined in Γ by a positive quantifier-free formula in the language of Γ-sets. Then $(\mathbf{k}, \Gamma)$ is a Mann pair.*

Next we improve Lemma 7.3.4. We shall use facts from [9], but alert the reader that in the statement and proof of Proposition 5.11 of [9], "subgroups of G^n" should be "cosets of subgroups of G^n", and "B_n" should be "B_m".

Proposition 7.3.5. *Suppose for each $n \geq 1$ the set $\Sigma_n(\mathbf{k}, \Gamma)$ is a finite union of sets $X \times Y$ where $X \subseteq \mathbf{k}^n$ and where $Y \subseteq \Gamma^n$ is a boolean combination of cosets of subgroups of Γ^n. Then $(\mathbf{k}, \Gamma)$ is a Mann pair.*

Proof. Let $n \geq 1$. For $(x, y) \in \mathbf{k}^n \times \Gamma^n$ we have

$$x_1y_1 + \cdots + x_ny_n = 1 \iff \big((x_1, \ldots, x_n, -1), (y_1, \ldots, y_n, 1)\big) \in \Sigma_{n+1}(\mathbf{k}, \Gamma).$$

Using the hypothesis with $n+1$ instead of n it follows easily that the set

$$\{(x, y) \in \mathbf{k}^n \times \Gamma^n : \ x_1y_1 + \cdots + x_ny_n = 1\}$$

is a finite union of sets $X \times Y$ where $X \subseteq \mathbf{k}^n$ and where $Y \subseteq \Gamma^n$ is a boolean combination of cosets of subgroups of Γ^n. Then Proposition 5.11 of [9] yields that Γ has the Mann property. Next,

$$\Sigma_n(\mathbf{k}, \Gamma) = \bigcup_{i \in I} X_i \times Y_i$$

with finite I and each $X_i \subseteq \mathbf{k}^n$, and where each $Y_i \subseteq \Gamma^n$ is a boolean combination of cosets of subgroups of Γ^n. We can also arrange that $X_i \neq \emptyset$ for all i and $X_i \cap X_j = \emptyset$ for all distinct $i, j \in I$.

By Corollary 5.1 and Lemma 5.5 in [9], it follows that each set Y_i is definable in Γ by a positive quantifier-free formula in the language of Γ-sets. Then Lemma 7.3.4 yields that $(\mathbf{k}, \Gamma)$ is a Mann pair. □

Lemma 7.3.6. *If* $(\mathbf{k}, \Gamma)$ *is a Mann pair, then it is a Mann pair over* $\mathbf{k}(\Gamma)$*.*

Proof. Assume $(\mathbf{k}, \Gamma)$ is a Mann pair, and let $r_1, \ldots, r_n \in \mathbf{k}(\Gamma)^\times$. By Lemma 7.3.3, it suffices to prove that $\Sigma(\vec{r}, \mathbf{k}, \Gamma)$ is a finite union of sets $P \times Q$, where $P \subseteq \mathbf{k}^n$, and $Q \subseteq \Gamma^n$ is defined by a finite conjunction of atoms in the language of Γ-sets. We may assume that $r_1, \ldots, r_n \in \mathbf{k}[\Gamma]$. Then $r_i = \sum_{j=1}^{k} a_{ij}\beta_j$ with $a_{ij} \in \mathbf{k}$, and $\beta_j \in \Gamma$ for $i = 1, \ldots, n$ and $j = 1, \ldots, k$.

Let $\vec{a} = (a_{11}, \ldots, a_{1k}, \ldots, a_{n1}, \ldots, a_{nk}) \in \mathbf{k}^{nk}$. Then

$$(c_1, \ldots, c_n, \gamma_1, \ldots, \gamma_n) \in \Sigma(\vec{r}, \mathbf{k}, \Gamma)$$

$$\iff$$

$$(c_1, \ldots, c_1, \ldots, c_n, \ldots, c_n, \beta_1\gamma_1, \ldots, \beta_k\gamma_1, \ldots, \beta_1\gamma_n, \ldots, \beta_k\gamma_n) \in \Sigma(\vec{a}, \mathbf{k}, \Gamma).$$

By Lemma 7.3.2, $\Sigma(\vec{a}, \mathbf{k}, \Gamma)$ is a finite union of sets $P \times Q$, where $P \subseteq \mathbf{k}^{nk}$, and $Q \subseteq \Gamma^{nk}$ is defined in Γ by a conjunction of atoms in the language of Γ-sets. Fix such P, Q, and define

$$P' := \{(c_1, \ldots, c_n) \in \mathbf{k}^n : \ (c_1, \ldots, c_1, \ldots, c_n, \ldots, c_n) \in P\},$$

$$Q' := \{(\gamma_1, \ldots, \gamma_n) \in \Gamma^n : \ (\beta_1\gamma_1, \ldots, \beta_k\gamma_1, \ldots, \beta_1\gamma_n, \ldots, \beta_k\gamma_n) \in Q\}.$$

Then for $(c_1,\ldots,c_n,\gamma_1,\ldots,\gamma_n) \in \mathbf{k}^n \times \Gamma^n$,

$$(c_1,\ldots,c_n,\gamma_1,\ldots,\gamma_n) \in P' \times Q'$$

$$\iff$$

$$(c_1,\ldots,c_1,\ldots,c_n,\ldots,c_n,\beta_1\gamma_1,\ldots,\beta_k\gamma_n) \in P \times Q.$$

Hence, $\Sigma(\vec{a},\mathbf{k},\Gamma)$ being a finite union of sets $P \times Q$ with P and Q as above, $\Sigma(\vec{r},\mathbf{k},\Gamma)$ is the union of the corresponding sets $P' \times Q'$. This finishes the proof of the lemma. ☐

Proposition 7.3.7. *If $(\mathbf{k},\Gamma)$ is a Mann pair, then it is a Mann pair over Ω.*

Proof. Let $\vec{r} = (r_1,\ldots,r_n) \in \Omega^n$. Take a basis $b_1,\ldots,b_m$ of the $\mathbf{k}(\Gamma)$-linear space $\mathbf{k}(\Gamma)r_1 + \cdots + \mathbf{k}(\Gamma)r_n$. Then $r_j = \sum_{i=1}^m r_{ij}b_i$ with $r_{ij} \in \mathbf{k}(\Gamma)$, so that for all $(\vec{c},\vec{\gamma}) \in \mathbf{k}^n \times \Gamma^n$ we have: $(\vec{c},\vec{\gamma}) \in \Sigma(\vec{r},\mathbf{k},\Gamma)$ if and only if $(c_1\gamma_1,\ldots,c_n\gamma_n)$ is a solution of the system

$$\begin{aligned}
r_{11}x_1 + \cdots + r_{1n}x_n &= 0\\
r_{21}x_1 + \cdots + r_{2n}x_n &= 0\\
&\vdots\\
r_{m1}x_1 + \cdots + r_{mn}x_n &= 0.
\end{aligned}$$

Suppose now that $(\mathbf{k},\Gamma)$ is a Mann pair. Then it is a Mann pair over $\mathbf{k}(\Gamma)$ by the previous lemma. It remains to use Lemmas 7.3.2 and 7.3.3. ☐

Clearly, this proposition and Lemma 7.3.2 yield Proposition 7.3.1.

For any n-tuple $k = (k_1,\ldots,k_n) \in \mathbb{Z}^n$, consider the *character*

$$\chi_k : \left(\Omega^\times\right)^n \to \Omega^\times, \quad \chi_k(y_1,\ldots,y_n) := y_1^{k_1}\cdots y_n^{k_n}.$$

This is a multiplicative group homomorphism. For any $e \in \mathbb{N}$, let $\mathcal{D}(n,e)$ be the finite collection of subgroups of $\left(\Omega^\times\right)^n$ that are intersections of kernels of characters χ_k with $|k| = |k_1| + \cdots + |k_n| \le e$.

Proposition 7.3.8. *Let $(\mathbf{k},\Gamma)$ be a Mann pair, let $X_1,\ldots,X_m,Y_1,\ldots,Y_n$ be distinct indeterminates, and let the polynomials $f_1,\ldots,f_M \in \Omega[X,Y]$ have degree at most d in $X = (X_1,\ldots,X_m)$ and degree at most e*

in $Y = (Y_1, \dots, Y_n)$*, where* $M, d, e \in \mathbb{N}$*, and put*

$$Z := \{(x, y) \in \mathbf{k}^m \times \Gamma^n : f_1(x, y) = \cdots = f_M(x, y) = 0\}.$$

Then Z *is a finite union of sets* $P \times Q$*, where*

$$P = \{x \in \mathbf{k}^m : g_1(x) = \cdots = g_N(x) = 0\}$$

for suitable $N \in \mathbb{N}$ *and polynomials* $g_1, \dots, g_N \in \mathbf{k}[X]$ *of degree* $\leq d$*, and where* $Q \subseteq \Gamma^n$ *is a coset of a subgroup* $D \cap \Gamma^n$ *of* Γ^n *with* $D \in \mathcal{D}(n, e)$*.*

Proof. The intersection of finitely many cosets of such subgroups is either empty or again a coset of such a subgroup. Hence we may (and shall) assume that $M = 1$. Put $f := f_1$. Then $f = \sum_{(i,j) \in I \times J} a_{ij} X^i Y^j$ where all $a_{ij} \in \Omega$ and I is the set of multi-indices $i = (i_1, \dots, i_m) \in \mathbb{N}^m$ with $|i| = i_1 + \cdots + i_m \leq d$ and J is the set of multi-indices $j = (j_1, \dots, j_n) \in \mathbb{N}^n$ with $|j| = j_1 + \cdots + j_n \leq e$. By Proposition 7.3.7 and Lemma 7.3.2 the set

$$\big\{\big((x_i)_{i \in I}, (y_j)_{j \in J}\big) \in \mathbf{k}^I \times \Gamma^J : \sum_{(i,j) \in I \times J} a_{ij} x_i y_j = 0\big\}$$

is a finite union of subsets of $\mathbf{k}^I \times \Gamma^J$ of the form

$$V \times \{y \in \Gamma^J : \ \gamma_1 y_{i(1)} = y_{j(1)}, \dots, \gamma_k y_{i(k)} = y_{j(k)}\}$$

with V a $\mathbf{k}$-linear subspace of $\mathbf{k}^I$, $\gamma_1, \dots, \gamma_k \in \Gamma$, and with $k \in \mathbb{N}$ and indices $i(1), j(1), \dots, i(k), j(k)$ in J. It remains to observe that for such $\gamma_1, \dots, \gamma_k, i(1), j(1), \dots, i(k), j(k)$ the set

$$\{y \in \Gamma^J : \ \gamma_1 \chi_{i(1)}(y) = \chi_{j(1)}(y), \dots, \gamma_k \chi_{i(k)}(y) = \chi_{j(k)}(y)\}$$

is a coset of the subgroup $D \cap \Gamma^n$ of Γ^n where D is the intersection of the kernels of $\chi_{i(1)-j(1)}, \dots, \chi_{i(k)-j(k)}$.

□

7.4 Robustness of Mann pairs

In this section we assume that $(\mathbf{k}, \Gamma)$ is a Mann pair. In addition, $K \supseteq \mathbf{k}$ is a subfield of Ω, and Δ, Γ' are subgroups of $\Omega^\times$ with $\Gamma' \supseteq \Gamma$. We claim:

(1) $[K : \mathbf{k}] < \infty \ \Rightarrow \ (K, \Gamma)$ is a Mann pair;

(2) $[\Gamma' : \Gamma] < \infty \;\Rightarrow\; (\mathbf{k}, \Gamma')$ is a Mann pair;

(3) $\big(K \supseteq \mathbf{k}(\Gamma)$ and (K, Δ) is a Mann pair$\big) \;\Rightarrow\; (\mathbf{k}, \Gamma\Delta)$ is a Mann pair;

(4) $\big(K$ is linearly disjoint from $\mathbf{k}(\Gamma)$ over $\mathbf{k}\big) \Rightarrow\; (K, \Gamma)$ is a Mann pair;

Proof of (1)**.** Here K is an extension field of finite degree m over $\mathbf{k}$. Let $b_1, \dots, b_m$ be a basis of the $\mathbf{k}$-linear space K. Let $n \geq 1$, and consider $(\vec{a}, \vec{\gamma}) \in K^n \times \Gamma^n$. Then with $a_j = \sum_{i=1}^{m} a_{ij} b_i$ $(1 \leq j \leq n,\;$ all $a_{ij} \in \mathbf{k})$,

$$\begin{aligned} (\vec{a}, \vec{\gamma}) \in \Sigma_n(K, \Gamma) &\iff \sum_{j=1}^{n} (a_{1j} b_1 + \dots + a_{mj} b_m)\gamma_j = 0 \\ &\iff F(a_{11}, \dots, a_{mn}, \gamma_1, \dots, \gamma_n) = 0 \\ \text{with } \; F := \sum_{j=1}^{n} &(X_{1j} b_1 + \dots + X_{mj} b_m) Y_j \in K[X_{11}, \dots, X_{mn}, Y_1, \dots, Y_n]. \end{aligned}$$

By Proposition 7.3.8, the set

$$\{\big((c_{ij}), \gamma\big) \in \mathbf{k}^{mn} \times \Gamma^n : F(c_{11}, \dots, c_{mn}, \gamma_1, \dots, \gamma_n) = 0\}$$

is a union $\bigcup_{i \in I} P_i \times Q_i$ with finite I, where each $P_i \subseteq \mathbf{k}^{mn}$, and each $Q_i \subseteq \Gamma^n$ is a coset of a subgroup of Γ^n. Therefore, $\Sigma_n(K, \Gamma) = \bigcup_{i \in I} P_i' \times Q_i$ with $P_i' = \{\vec{a} \in K^n : (a_{ij}) \in P_i\}$, where $\vec{a}$ determines (a_{ij}) as above.

Corollary 7.4.1. *If $\gamma \in \Gamma$ is algebraic over $\mathbf{k}$, then γ is a root of unity. In particular, if $(\mathbb{Q}, \Gamma)$ is a Mann pair with $\Gamma \subseteq A^\times$, then $\Gamma \subseteq \mathbb{U}$.*

Proof of (2)**.** Assume $\Gamma' = \bigcup_{\alpha \in C} \alpha\Gamma$ where $C \subseteq \Gamma'$ is finite. Then

$$\Sigma_n(\mathbf{k}, \Gamma') = \bigcup_{\vec{\alpha} \in C^n} \{(\vec{a}, \vec{\alpha}\vec{\gamma}) : \; (\vec{a}, \vec{\gamma}) \in \Sigma(\vec{\alpha}, \mathbf{k}, \Gamma),$$

and it remains to use Proposition 7.3.7, Lemma 7.3.2 and Proposition 7.3.5.

Proof of (3)**.** Assume $K \supseteq \mathbf{k}(\Gamma)$ and (K, Δ) is a Mann pair. Let $n \geq 1$ and take finite $\Delta(n) \subseteq \Delta^n$ such that

$$\Sigma_n^{\mathrm{nd}}(K, \Delta) = \bigcup_{\vec{\delta} \in \Delta(n)} \Sigma_n^{\mathrm{nd}}(K, \Delta; \vec{\delta}) \times \Delta\vec{\delta}.$$

For each $\vec{\delta} \in \Delta(n)$ we have a finite $\Gamma(\vec{\delta}) \subseteq \Gamma^n$ such that

$$\Sigma^{\mathrm{nd}}(\vec{\delta}, \mathbf{k}, \Gamma) = \bigcup_{\vec{\gamma} \in \Gamma(\vec{\delta})} \Sigma^{\mathrm{nd}}(\vec{\delta}, \mathbf{k}, \Gamma; \vec{\gamma}) \times \Gamma\vec{\gamma}.$$

It follows that, with $\vec{\gamma}\vec{\delta} := (\gamma_1\delta_1, \ldots, \gamma_n\delta_n)$, we have

$$\Sigma_n^{\mathrm{nd}}(\mathbf{k}, \Gamma\Delta) = \bigcup_{\vec{\delta}\in\Delta(n)} \bigcup_{\vec{\gamma}\in\Gamma(\vec{\delta})} \Sigma_n^{\mathrm{nd}}(\mathbf{k}, \Gamma\Delta; \vec{\gamma}\vec{\delta}) \times \Gamma\Delta\vec{\gamma}\vec{\delta}.$$

Proof of (4)**.** Assume K is linearly disjoint from $\mathbf{k}(\Gamma)$ over $\mathbf{k}$. Let $n \geq 1$; it is enough to obtain $\Sigma_n(K, \Gamma)$ as a finite union of sets $P \times Q$ with $P \subseteq K^n$, and $Q \subseteq \Gamma^n$ defined in Γ by a conjunction of atoms in the language of Γ-sets.

Suppose $(\vec{r}, \vec{\gamma}) \in \Sigma_n(K, \Gamma)$. Take a basis $b_1, \ldots, b_m$ of the $\mathbf{k}$-linear space $\mathbf{k}r_1 + \cdots + \mathbf{k}r_n$. Then

$$r_j = \sum_{i=1}^{m} a_{ij} b_i$$

with $a_{ij} \in \mathbf{k}$, $j = 1, \ldots, n$, hence

$$(a_{i1}, \ldots, a_{in}, \vec{\gamma}) \in \Sigma_n(\mathbf{k}, \Gamma) \quad \text{for } i = 1, \ldots, m.$$

Conversely, given any $a_{ij} \in \mathbf{k}$ $(1 \leq i, j \leq n)$ such that

$$(a_{i1}, \ldots, a_{in}, \vec{\gamma}) \in \Sigma_n(\mathbf{k}, \Gamma) \quad \text{for } i = 1, \ldots, n$$

and any $b_1, \ldots, b_n \in K$ (not necessarily linearly independent over $\mathbf{k}$), we have $(\vec{r}, \vec{\gamma}) \in \Sigma_n(K, \Gamma)$, where $r_j = \sum_{i=1}^{n} a_{ij} b_i$ for $j = 1, \ldots, n$. It follows that for all $(\vec{r}, \vec{\gamma}) \in K^n \times \Gamma^n$ we have: $(\vec{r}, \vec{\gamma}) \in \Sigma_n(K, \Gamma)$ if and only if there is an $n \times n$-matrix $A = (a_{ij}) \in \mathbf{k}^{n^2}$ and a vector $\vec{b} \in K^n$ such that

$$(a_{i1}, \ldots, a_{in}, \vec{\gamma}) \in \Sigma_n(\mathbf{k}, \Gamma) \quad \text{for } i = 1, \ldots, n, \text{ and } A\vec{b} = \vec{r}.$$

with $\vec{b}$ and $\vec{r}$ viewed as column vectors. By Lemma 7.3.2 the set of all $(A, \vec{\gamma}) \in \mathbf{k}^{n^2} \times \Gamma^n$, with $A = (a_{ij})$, such that the displayed condition holds, is a union $\bigcup_{\lambda\in\Lambda} P_\lambda \times Q_\lambda$ with finite Λ where each $P_\lambda \subseteq \mathbf{k}^{n^2}$, and each $Q_\lambda \subseteq \Gamma^n$ is defined in Γ by a conjunction of atoms in the language of Γ-sets.

For $\lambda \in \Lambda$, let P'_λ be the set of all $\vec{r} \in K^n$ for which there is a matrix $A \in P_\lambda$ and a vector $\vec{b} \in K^n$ such that $A\vec{b} = \vec{r}$. Then by the above, $\Sigma_n(K, \Gamma) = \bigcup_{\lambda\in\Lambda} P'_\lambda \times Q_\lambda$.

7.4.1 Curious Mann pairs in positive characteristic

Let $\Omega \supseteq \mathbb{F}_p$ be of characteristic p so that every $\mathbb{F}_{p^e}$ with e a positive integer is a subfield of Ω. Let P be a finite nonempty set of prime numbers different from p, let $S(P)$ be the set of all positive integers all of whose prime factors are in P. Put

$$G := \{a \in \Omega^\times : \ a^N = 1 \text{ for some } N \in S(P)\},$$

an infinite subgroup of $\Omega^\times$ isomorphic to the direct product of the Prüfer groups $\mathbb{Z}(l^\infty)$ with $l \in P$.

Theorem 8.9 in [9] says that G has the Mann property, but this is also a consequence of earlier results in [5] and [24]. We now strengthen this as follows. The proof in [9] starts with a certain large enough finite field $\mathbb{F}_{p^e}$ where e is a positive integer. Put

$$f := e \cdot \prod_{l \notin P} l^\infty, \quad g := e \cdot \prod_{l \in P} l^\infty \qquad \text{(products of supernatural numbers)}.$$

Then $\mathbb{F}_{p^e}(G) = \mathbb{F}_{p^g} \subseteq \Omega$, and so the infinite field $\mathbb{F}_{p^f}$ is linearly disjoint from $\mathbb{F}_{p^e}(G)$ over $\mathbb{F}_{p^e}$, so by (4) above we have a Mann pair $(\mathbb{F}_{p^f}, G)$ with the curious property that $\mathbb{F}_{p^f}(G) = \mathbb{F}_p^{\text{ac}}$.

7.4.2 Some Mann pairs in mixed characteristic

Let $E \supseteq \mathbb{F}_p$ be a perfect field of characteristic p. Let $W[E] \supseteq W[\mathbb{F}_p] = \mathbb{Z}_p$ be the ring of Witt vectors over E, let $W(E) \supseteq W(\mathbb{F}_p) = \mathbb{Q}_p$ be its fraction field, and let $\tau : E^\times \to W(E)^\times$ be the Teichmüller lifting. It was shown in [9], 8.4, that $\tau(E^\times)$ satisfies a strong form of the Mann property. In fact:

Proposition 7.4.2. *$\big(\mathbb{Q}_p, \tau(E^\times)\big)$ is a Mann pair.*

Proof. Consider first the case $E = \mathbb{F}_p^{\text{ac}}$. Then

$$\tau(E^\times) = \mathbb{U}[p'] := \{x \in W(E) : \ x^n = 1 \text{ for some } n \text{ not divisible by } p\}.$$

By arguments as in the proof of Theorem 1 in [17] one shows that $(\mathbb{Q}_p, \mathbb{U}[p'])$ is a Mann pair. (Alternatively, $(\mathbb{Q}, \mathbb{U}[p'])$ is a Mann pair by Theorem 1 in [17], and it remains to use (4) and the fact that $\mathbb{Q}_p$ and $\mathbb{Q}(\mathbb{U}[p'])$ are linearly disjoint over $\mathbb{Q}$.)

In the general case, we first arrange that E is algebraically closed so that $E \supseteq \mathbb{F}_p^{\text{ac}}$ and $W(E) \supseteq W(\mathbb{F}_p^{\text{ac}})$. Then $E^\times = (\mathbb{F}_p^{\text{ac}})^\times G$ where the subgroup G of $E^\times$ is torsion-free. Then $\tau(E^\times) = \mathbb{U}[p']\tau(G)$. By the proof of 8.4 in [9], whenever $\gamma_1, \dots, \gamma_n \in \tau(G)$ are multiplicatively independent, they are algebraically independent over $W(\mathbb{F}_p^{\mathrm{ac}})$. Then Lemma 7.1.3 yields that $\big(W(\mathbb{F}_p^{\mathrm{ac}}), \tau(G)\big)$ is a Mann pair. Since $(\mathbb{Q}_p, \mathbb{U}[p'])$ is a Mann pair, we obtain from (3) that $(\mathbb{Q}_p, \tau(E^\times))$ is a Mann pair. $\square$

7.5 Proof of Theorem 1.0.3

In this section we assume that Ω has characteristic zero, and, besides $\mathbf{k}$, we also let E, F, K be subfields of Ω. The following contains Proposition 5.16 of [9] as a special case, with almost the same proof.

Lemma 7.5.1. *Let E be a subfield of K such that all p^{th} roots of unity in K are in E, for every p. Let G be a pure subgroup of $E^\times$, and put*

$$H := \{h \in K^\times : \ h^d \in G \textit{ for some positive integer } d\}.$$

Then for $a_1, \ldots, a_n \in E^\times$, the equation $a_1x_1 + \cdots + a_nx_n = 1$ has the same non-degenerate solutions in G as in H.

Proof. By Lemma 5.15 of [9], it is enough to show the following.

Claim. Let $h \in K^\times$ be such that $h^p = g \in G$ and $h \notin G$. Then $X^p - g \in E[X]$ is irreducible, and $Gh^{\mathbb{Z}}$, the subgroup of $K^\times$ generated by G and h, is pure in $E(h)^\times$.

Proof of the Claim. Since $h \notin G$ and K and E have the same p^{th} roots of unity, g is not a p^{th} power in $E^\times$. Thus by Theorem 9.1 from [15] the polynomial $X^p - g$ is irreducible in $E[X]$. To show that $Gh^{\mathbb{Z}} = \bigcup_{i=0}^{p-1} Gh^i$ is pure in $E(h)^\times$, suppose towards a contradiction that $f \in E(h)^\times$ and $f^d \in Gh^{\mathbb{Z}}$ where d is an integer greater than 1, but $f \notin Gh^{\mathbb{Z}}$. We can reduce to the case that d is prime and $f^m \notin G$ for $1 \le m < d$. So by Theorem 9.1 of [15] again, $X^d - f^d$ is irreducible in $E[X]$. As $E(f) \subseteq E(h)$ and $[E(h) : E] = p$, we get $d = p$. Let ζ be a primitive p^{th} root of unity in the algebraic closure of K. Then $E(h) \cap E(\zeta) = E$ and $E(h,\zeta)$ is a cyclic extension of degree p of $E(\zeta)$. Let $\sigma \in \mathrm{Gal}\big(E(h,\zeta)|E(\zeta)\big)$ be given by $\sigma(h) = \zeta h$. Then $\sigma(f) = \zeta^k f$ with $0 < k < p$. With $f = c_0 + c_1h + \cdots + c_{p-1}h^{p-1}$ and all $c_i \in E$ this gives

$$\begin{aligned}\sigma(f) &= c_0 + \zeta c_1 h + \cdots + \zeta^{p-1} c_{p-1} h^{p-1} \\ &= \zeta^k c_0 + \zeta^k c_1 h + \cdots + \zeta^k c_{p-1} h^{p-1}.\end{aligned}$$

This forces $c_i = 0$ for all $i \neq k$, so $f = c_k h^k$, as desired. $\square$

Lemma 7.5.2. *Suppose $\mathbf{k}$ is algebraically closed, $F \supseteq \mathbf{k}$, $\mathrm{trdeg}(F|\mathbf{k}) = 1$, $\mathbf{k}^\times \cap \Gamma = \{1\}$, and $\Gamma \subseteq F^\times$ has finite rank. Then $(\mathbf{k}, \Gamma)$ is a Mann pair.*

Proof. We can assume that $\Gamma \neq \{1\}$. Consider the divisible closure

$$H := \{f \in F^{\times} :\ f^d \in \mathbf{k}^{\times}\Gamma \text{ for some positive integer } d\},$$

of $\mathbf{k}^{\times}\Gamma$ in $F^{\times}$. Now, as an abelian group, $\mathbf{k}^{\times}$ is divisible and hence injective, so we can take a subgroup $\Gamma' \supseteq \Gamma$ of H such that $\mathbf{k}^{\times} \cap \Gamma' = \{1\}$ and $\mathbf{k}^{\times}\Gamma' = H$. It follows that Γ' is pure in $F^{\times}$ and Γ'/Γ is a torsion group. Replacing Γ by Γ' we arrange that Γ is pure in $F^{\times}$. Take a finitely generated subgroup Γ_0 of Γ such that Γ/Γ_0 is a torsion group. Then $E := \mathbf{k}(\Gamma_0)$ is a function field over $\mathbf{k}$ of one variable. Take a finite subset S of $\mathcal{R}(E|\mathbf{k})$ such that all elements of Γ_0 are S-units, and put $\Gamma_1 := E^{\times} \cap \Gamma$. Since Γ_1/Γ_0 is a torsion group, all elements of Γ_1 are also S-units. Hence Γ_1 is isomorphic to a subgroup of $\bigoplus_{v \in S} \mathbb{Z}v$, and is thus finitely generated. Then by Corollary 7.2.2, $(\mathbf{k}, \Gamma_1)$ is a Mann pair. Since Γ is pure in $F^{\times}$ and $\mathbf{k}$ contains all roots of unity, Γ_1 is pure in $E^{\times}$. Now apply Lemmas 7.5.1 and 7.1.1 to get that $(\mathbf{k}, \Gamma)$ is a Mann pair. $\square$

We are ready to prove Theorem 1.0.3 from the Introduction:

Theorem 7.5.3. *Suppose* $\mathbf{k}$ *is algebraically closed,* $\mathbf{k}^{\times} \cap \Gamma = \{1\}$*, and* Γ *has finite rank. Then* $(\mathbf{k}, \Gamma)$ *is a Mann pair.*

Proof. Let F be the algebraic closure of $\mathbf{k}(\Gamma)$ in Ω, and put

$$H := \{f \in F^{\times} :\ f^d \in \mathbf{k}^{\times}\Gamma \text{ for some positive integer } d\},$$

the divisible closure of $\mathbf{k}^{\times}\Gamma$ in $F^{\times}$. Now, as an abelian group, $\mathbf{k}^{\times}$ is divisible and hence injective, so we can take a subgroup $\Gamma' \supseteq \Gamma$ of H such that $\mathbf{k}^{\times} \cap \Gamma' = \{1\}$ and $\mathbf{k}^{\times}\Gamma' = H$. It follows that Γ' is divisible and Γ'/Γ is a torsion group. So replacing Γ by Γ' if necessary we can assume in the rest of the proof that Γ is divisible. Let $m := \operatorname{trdeg}(\mathbf{k}(\Gamma)|\mathbf{k})$, and take a chain

$$\mathbf{k} = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_m = F$$

of algebraically closed subfields of Ω such that $\operatorname{trdeg}(K_{i+1}|K_i) = 1$ for $i = 0, \ldots, m-1$. Put

$$\Gamma_i := \Gamma \cap K_i^{\times}, \quad i = 1, \ldots, m.$$

We claim that $(\mathbf{k}, \Gamma_i)$ is a Mann pair for $i = 1, \ldots, m$. For $i = 1$ this holds by Lemma 7.5.2, since $\mathbf{k}(\Gamma_1) \subseteq K_1$ and K_1 has transcendence degree 1 over $\mathbf{k}$. Suppose $1 \leq i < m$ and $(\mathbf{k}, \Gamma_i)$ is a Mann pair. Since Γ_i is divisible we have a subgroup Δ of Γ_{i+1} such that $\Gamma_{i+1} = \Gamma_i\Delta$ and $\Gamma_i \cap \Delta = \{1\}$. Then Δ has finite rank and $K_i^\times \cap \Delta = \{1\}$, and $K_i(\Delta) \subseteq K_{i+1}$, so (K_i, Δ) is a Mann pair. It follows from (3) in Section 5 that then $(\mathbf{k}, \Gamma_i\Delta) = (\mathbf{k}, \Gamma_{i+1})$ is a Mann pair.

This finishes the inductive proof of the claim. For $i = m$ this claim yields the desired conclusion. $\square$

In this theorem we cannot replace the finite rank assumption by the weaker condition that Γ has the Mann property, as the following example shows. Take distinct indeterminates $a_1, x_1, a_2, x_2, a_3, x_3, \dots$ and put

$$\mathbf{k} := \mathbb{Q}(a_1, a_2, a_3, \dots)^{\mathrm{ac}}, \qquad K := \mathbf{k}(x_1, x_2, x_2, \dots).$$

Let Γ be the subgroup of $K^\times$ generated by $x_1, y_1, x_2, y_2, x_3, y_3, \dots$ where $y_i := 1 - a_i x_i$. Then $\mathbf{k}$ is algebraically closed, $\mathbf{k}^\times \cap \Gamma = \{1\}$, but $(\mathbf{k}, \Gamma)$ is not a Mann pair, since $a_i x_i + y_i = 1$ for all i. On the other hand, Γ has the Mann property because $x_1, y_1, x_2, y_2, \dots$ are algebraically independent over $\mathbb{Q}$ (but not over $\mathbf{k}$).

The condition $\mathbf{k}^\times \cap \Gamma = \{1\}$ in the theorem can be relaxed:

Corollary 7.5.4. *Suppose $\mathbf{k}$ is algebraically closed, $\mathbf{k}^\times \cap \Gamma$ is finite and Γ has finite rank. Then $(\mathbf{k}, \Gamma)$ is a Mann pair.*

Proof. Put $d := |\mathbf{k}^\times \cap \Gamma|$. The hypothesis implies that then the subgroup $\Delta := \{\gamma^d : \gamma \in \Gamma\}$ of Γ is of finite rank and has finite index in Γ, and that $\mathbf{k}^\times \cap \Delta = \{1\}$. Hence by the theorem, $(\mathbf{k}^{\mathrm{ac}}, \Delta)$ is a Mann pair, and thus $(\mathbf{k}, \Gamma)$ is a Mann pair by item (2) of the previous section. $\square$

7.6 Definable relations in Mann pairs

In this section we characterize Mann pairs model-theoretically:

Theorem 7.6.1. *The following are equivalent:*

1. *$(\mathbf{k}, \Gamma)$ is a Mann pair;*
2. *for each $n \geq 1$ the set $\{(x, y) \in \mathbf{k}^n \times \Gamma^n : x_1y_1 + \cdots + x_ny_n = 0\}$ is a finite union of sets $X \times Y$ where $X \subseteq \mathbf{k}^n$ is definable in the field $\mathbf{k}$ and $Y \subseteq \Gamma^n$ is definable in the group Γ;*
3. *for all m, n, every subset of $\mathbf{k}^m \times \Gamma^n$ definable in $(\Omega, \mathbf{k}, \Gamma)$ is a finite union of sets $X \times Y$ with $X \subseteq \mathbf{k}^m$ definable in the field $\mathbf{k}$ and $Y \subseteq \Gamma^n$ definable in the group Γ.*

Let $\mathcal{L}$ be the language of rings augmented by two distinct unary relation symbols. Let T be the $\mathcal{L}$-theory whose models are the structures $(\Omega, \mathbf{k}, \Gamma)$ where Ω is an algebraically closed field with a subfield $\mathbf{k}$, and a subgroup Γ of $\Omega^\times$. Let $\mathcal{L}^{\mathrm{f,g}}_{\Sigma}$ be the 2-sorted language, with sorts f, g (the *field sort* and the *group sort*), and with the following nonlogical symbols:

- constant symbols 0 and 1 of sort f,
- a unary function symbol $-$ of sort $(\mathrm{f};\mathrm{f})$,
- binary function symbols $+$ and $\cdot$ of sort $(\mathrm{f},\mathrm{f};\mathrm{f})$,
- a constant symbol 1 of sort g,
- a unary function symbol $^{-1}$ of sort $(\mathrm{g};\mathrm{g})$
- a binary function symbol $\cdot$ of sort $(\mathrm{g},\mathrm{g};\mathrm{g})$,
- for each $n \geq 1$, a $2n$-ary relation symbol Σ_n of sort $(\mathrm{f},\dots,\mathrm{f},\mathrm{g},\dots,\mathrm{g})$ (n places of sort f and n places of sort g).

Of course, "1 used as a symbol of sort f" is different from "1 used as a symbol of sort g" and likewise with the multiplication symbol. For a model $(\Omega, \mathbf{k}, \Gamma)$ of T we construe $(\mathbf{k}, \Gamma)$ as an $\mathcal{L}^{\mathrm{f,g}}_{\Sigma}$-structure by interpreting the symbols in the obvious way; in particular, each Σ_n is interpreted as the previously defined $\Sigma_n(\mathbf{k},\Gamma) \subseteq \mathbf{k}^n \times \Gamma^n$.

In the next result we do *not* assume that $(\mathbf{k}, \Gamma)$ is a Mann pair.

Lemma 7.6.2. *Let* $(\Omega, \mathbf{k}, \Gamma) \models T$. *Then every subset of* $\mathbf{k}^m \times \Gamma^n$ *definable in* $(\Omega, \mathbf{k}, \Gamma)$ *is definable in the* $\mathcal{L}^{\mathrm{f,g}}_{\Sigma}$*-structure* $(\mathbf{k}, \Gamma)$.

Proof. Take an $|\Omega|^+$-saturated elementary extension $(\Omega', \mathbf{k}', \Gamma')$ of $(\Omega, \mathbf{k}, \Gamma)$. It is easy to check that then $\mathbf{k}'(\Gamma')$ and Ω are linearly disjoint over $\mathbf{k}(\Gamma)$:

$$\begin{tikzcd}
 & & \Omega' \\
 & \mathbf{k}'(\Gamma') & \Omega \\
\mathbf{k}' & \mathbf{k}(\Gamma) & \\
\mathbf{k} & &
\end{tikzcd}$$

Let $\vec{a}, \vec{b} \in (\mathbf{k}')^m$ and $\vec{\alpha}, \vec{\beta} \in (\Gamma')^n$ be such that

$$\mathrm{tp}_{(\mathbf{k}',\Gamma')}\big((\vec{a},\vec{\alpha})|(\mathbf{k},\Gamma)\big) \;=\; \mathrm{tp}_{(\mathbf{k}',\Gamma')}\big((\vec{b},\vec{\beta})|(\mathbf{k},\Gamma)\big).$$

It suffices to prove that then

$$\mathrm{tp}_{(\Omega',\mathbf{k}',\Gamma')}\big((\vec{a},\vec{\alpha})|(\Omega,\mathbf{k},\Gamma)\big) \;=\; \mathrm{tp}_{(\Omega',\mathbf{k}',\Gamma')}\big((\vec{b},\vec{\beta})|(\Omega,\mathbf{k},\Gamma)\big).$$

The assumption on $(\vec{a},\vec{\alpha})$ and $(\vec{b},\vec{\beta})$ gives an automorphism of $(\mathbf{k}',\Gamma')$ over $(\mathbf{k},\Gamma)$ that takes $(\vec{a},\vec{\alpha})$ to $(\vec{b},\vec{\beta})$. This automorphism preserves the Σ-relations, so it extends to a field automorphism of $\mathbf{k}'(\Gamma')$ over $\mathbf{k}(\Gamma)$, which extends further to a field automorphism of Ω' over Ω by linear disjointness of $\mathbf{k}'(\Gamma')$ and Ω over $\mathbf{k}(\Gamma)$. This yields the desired equality of types. $\square$

The implication (3) $\Rightarrow$ (2) of Theorem 7.6.1 is trivial, so in view of Lemma 7.3.2 and Proposition 7.3.5, this theorem will be established once we prove the part (1) $\Rightarrow$ (3), which is the next result.

Proposition 7.6.3. *Suppose* $(\Omega,\mathbf{k},\Gamma)\models T$ *and* $(\mathbf{k},\Gamma)$ *is a Mann pair. Then every subset of* $\mathbf{k}^m\times\Gamma^n$ *that is definable in* $(\Omega,\mathbf{k},\Gamma)$ *is a finite union of sets* $X\times Y$ *with* $X\subseteq\mathbf{k}^m$ *definable in the field* $\mathbf{k}$ *and* $Y\subseteq\Gamma^n$ *definable in the group* Γ.

Proof. As in the proof of the previous lemma we take an $|\Omega|^+$-saturated elementary extension $(\Omega',\mathbf{k}',\Gamma')$ of $(\Omega,\mathbf{k},\Gamma)$ and tuples $\vec{a},\vec{b}\in(\mathbf{k}')^m$ and $\vec{\alpha},\vec{\beta}\in(\Gamma')^n$ such that

$$\mathrm{tp}_{\mathbf{k}'}(\vec{a}|\mathbf{k}) \;=\; \mathrm{tp}_{\mathbf{k}'}(\vec{b}|\mathbf{k}) \text{ and } \mathrm{tp}_{\Gamma'}(\vec{\alpha}|\Gamma) \;=\; \mathrm{tp}_{\Gamma'}(\vec{\beta}|\Gamma).$$

By Lemma 7.6.2 it is enough to show that then

$$\mathrm{tp}_{(\mathbf{k}',\Gamma')}\big((\vec{a},\vec{\alpha})|(\mathbf{k},\Gamma)\big) \;=\; \mathrm{tp}_{(\mathbf{k}',\Gamma')}\big((\vec{b},\vec{\beta})|(\mathbf{k},\Gamma)\big).$$

The assumption on $\vec{a}$ and $\vec{b}$ gives an automorphism σ of $\mathbf{k}'$ over $\mathbf{k}$ such that $\sigma(\vec{a})=\vec{b}$, and the assumption on $\vec{\alpha}$ and $\vec{\beta}$ gives an automorphism ϕ of Γ' over Γ such that $\phi(\vec{\alpha})=\vec{\beta}$. It remains to show that this gives an automorphism (σ,ϕ) of the $\mathcal{L}_{\Sigma}^{\mathrm{f,g}}$-structure $(\mathbf{k}',\Gamma')$. This in turn reduces to establishing the following: Let N be a positive integer and $\vec{c}\,'\in(\mathbf{k}')^N$ and $\vec{\gamma}'\in(\Gamma')^N$. Then

$$(\mathbf{k}',\Gamma')\models\Sigma_N(\vec{c}\,',\vec{\gamma}') \iff (\mathbf{k}',\Gamma')\models\Sigma_N(\sigma(\vec{c}\,'),\phi(\vec{\gamma}')).$$

We prove the forward implication. (The backward implication follows in the same way.) Suppose that $(\mathbf{k}',\Gamma')\models\Sigma_N(\vec{c}\,',\vec{\gamma}')$. We have $\Sigma_N(\mathbf{k},\Gamma)=\bigcup_{i\in I}P_i\times Q_i$ where I is finite, each $P_i\subseteq\mathbf{k}^N$ is definable in the field $\mathbf{k}$ and each $Q_i\subseteq\Gamma^N$ is definable in the group Γ. Then $\Sigma_N(\mathbf{k}',\Gamma')=\bigcup_{i\in I}P_i'\times Q_i'$ where $P_i'\subseteq(\mathbf{k}')^N$ is defined in $\mathbf{k}'$ by any formula with parameters from $\mathbf{k}$ that defines P_i in the field $\mathbf{k}$, and $Q_i'\subseteq(\Gamma')^N$ is defined in Γ' by any formula with parameters from Γ that defines Q_i in the group Γ. Take $i \in I$ such that $\vec{c}\,' \in P_i'$ and $\vec{\gamma}' \in Q_i'$. It is clear that then $\sigma(\vec{c}\,') \in P_i'$ and $\phi(\vec{\gamma}') \in Q_i'$, so $(\mathbf{k}', \Gamma') \models \Sigma_N(\sigma(\vec{c}\,'), \phi(\vec{\gamma}'))$, as desired. $\square$

7.7 The elementary theory of Ω with a Mann pair

In this section we discuss various model-theoretic properties of models $(\Omega, \mathbf{k}, \Gamma)$ of T where $(\mathbf{k}, \Gamma)$ is a Mann pair.

7.7.1 Elementary equivalence and smallness

We first prove that the theory of a model $(\Omega, \mathbf{k}, \Gamma)$ of T is determined by the $\mathcal{L}_\Sigma^{\mathrm{f,g}}$-theory of $(\mathbf{k}, \Gamma)$ whenever $\mathbf{k} \cup \Gamma$ is small in Ω. For this we do not need $(\mathbf{k}, \Gamma)$ to be a Mann pair. (A definition and basic properties of *small* are in [9], Section 2 and also in Section 3.1 of this thesis.)

Lemma 7.7.1. *Let* $(\Omega_1, \mathbf{k}_1, \Gamma_1)$ *and* $(\Omega_2, \mathbf{k}_2, \Gamma_2)$ *be models of* T *such that* $\mathbf{k}_i \cup \Gamma_i$ *is small in* Ω_i *for* $i = 1, 2$, *and* $(\mathbf{k}_1, \Gamma_1) \equiv (\mathbf{k}_2, \Gamma_2)$ *as* $\mathcal{L}_\Sigma^{\mathrm{f,g}}$*-structures. Then* $(\Omega_1, \mathbf{k}_1, \Gamma_1) \equiv (\Omega_2, \mathbf{k}_2, \Gamma_2)$.

Proof. It is harmless to assume CH (the continuum hypothesis), so we can reduce to the case that $(\Omega_i, \mathbf{k}_i, \Gamma_i)$ is saturated of cardinality $\aleph_1$ for $i = 1, 2$. Then we have an $\mathcal{L}_\Sigma^{\mathrm{f,g}}$-isomorphism

$$(\iota_{\mathrm{f}}, \iota_{\mathrm{g}}) : (\mathbf{k}_1, \Gamma_1) \to (\mathbf{k}_2, \Gamma_2).$$

This yields a ring isomorphism $\mathbf{k}_1[\Gamma_1] \to \mathbf{k}_2[\Gamma_2]$ extending both ι_{f} and ι_{g}, and hence a field isomorphism

$$\iota : \mathbf{k}_1(\Gamma_1)^{\mathrm{ac}} \to \mathbf{k}_2(\Gamma_2)^{\mathrm{ac}}.$$

By the smallness assumption the transcendence degree of Ω_i over $\mathbf{k}_i(\Gamma_i)^{\mathrm{ac}}$ is $\aleph_1$ for $i = 1, 2$. Thus we can extend ι to a field isomorphism $\Omega_1 \to \Omega_2$, which is then an $\mathcal{L}$-isomorphism. $\square$

Let $\mathbf{k}$ be a subfield of the algebraically closed field Ω. A theorem of E. Artin says that if $1 < [\Omega : \mathbf{k}] < \infty$, then $\mathbf{k}$ is real closed with $\Omega = \mathbf{k}(\sqrt{-1})$. As noted at the end of Section 2 in [9] it follows that by work of Keisler [13] the following are equivalent:

(1) $\mathbf{k}$ is small in Ω;

(2) $[\Omega : \mathbf{k}] > 2$;

(3) $[\Omega : \mathbf{k}] = \infty$.

We use this equivalence in the proof of the next lemma, which is analogous to Lemma 6.1 in [9]. (The proofs are not at all similar.)

Lemma 7.7.2. *Assume* $(\Omega, \mathbf{k}, \Gamma) \models T$ *where* $[\Omega : \mathbf{k}] > 2$ *and* $(\mathbf{k}, \Gamma)$ *is a Mann pair. Then* $\mathbf{k} \cup \Gamma$ *is small in* Ω.

Proof. We can assume that Γ is infinite. Take a proper elementary extension $(\Omega^*, \mathbf{k}^*, \Gamma^*)$ of $(\Omega, \mathbf{k}, \Gamma)$. Using that $(\mathbf{k}, \Gamma)$ is a Mann pair, it follows easily that the subfields $\mathbf{k}^*$ and $\mathbf{k}(\Gamma^*)$ of Ω^* are linearly disjoint over $\mathbf{k}$. See the diagram below where all arrows are inclusions:

$$\begin{tikzcd}
 & & \mathbf{k}^*(\Gamma^*) \\
 & \mathbf{k}^*(\Gamma) & \mathbf{k}(\Gamma^*) \\
\mathbf{k}^* & \mathbf{k}(\Gamma) & \\
\mathbf{k} & &
\end{tikzcd}$$

It follows that $\mathbf{k}^*(\Gamma)$ and $\mathbf{k}(\Gamma^*)$ are linearly disjoint over $\mathbf{k}(\Gamma)$. Among the subsets of Γ^* that are multiplicatively independent over Γ, take one, say B, that is maximal. It follows from Lemma 5.12 in [9] that B is algebraically independent over $\mathbf{k}(\Gamma)$. Hence B is algebraically independent over $\mathbf{k}^*(\Gamma)$. The maximality property of B guarantees that $B \neq \emptyset$ and for every $\gamma \in \Gamma^*$ there is a positive integer d with $\gamma^d \in \mathbf{k}^*(\Gamma)(B)$. Since $\mathbf{k}^*(\Gamma)(B)$ is a non-trivial purely transcendental extension of $\mathbf{k}^*(\Gamma)$ it follows that $\mathbf{k}^*(\Gamma^*, \sqrt{-1})$ is not algebraically closed, and so $\mathbf{k}^*(\Gamma^*, \sqrt{-1}) \neq \Omega^*$. Hence by the remark preceding this lemma $\mathbf{k}^* \cup \Gamma^*$ is small in Ω^*, and thus $\mathbf{k} \cup \Gamma$ is small in Ω; see Section 2 in [9]. $\square$

7.7.2 Stability

Proposition 7.7.3. *Let* $(\Omega, \mathbf{k}, \Gamma) \models T$ *be such that* $\mathbf{k}$ *is algebraically closed and* $(\mathbf{k}, \Gamma)$ *is a Mann pair. Then* $(\Omega, \mathbf{k}, \Gamma)$ *is stable, and if* Γ *is divisible, then* $(\Omega, \mathbf{k}, \Gamma)$ *is* ω*-stable.*

Proof. Take an infinite cardinal κ such that the abelian group Γ is κ-stable. We show that then $(\Omega, \mathbf{k}, \Gamma)$ is κ-stable. We can assume $\mathbf{k} \neq \Omega$, and that $|\Omega| = \kappa$. Take a κ^+-saturated elementary extension $(\Omega', \mathbf{k}', \Gamma')$ of $(\Omega, \mathbf{k}, \Gamma)$.

By the proofs of Lemma 7.6.2 and Proposition 7.6.3, the type of an element of $\mathbf{k}'$ over Ω in $(\Omega', \mathbf{k}', \Gamma')$ is determined by its type over $\mathbf{k}$ in the field $\mathbf{k}'$. Likewise, the type of an element of Γ' over Ω in $(\Omega', \mathbf{k}', \Gamma')$ is determined by its type over Γ in the group Γ'.

Now let $t \in \Omega(\mathbf{k}' \cup \Gamma')^{\mathrm{ac}}$, say $t \in \Omega(\vec{a}, \vec{\gamma})^{\mathrm{ac}}$ with $\vec{a} \in (\mathbf{k}')^m$ and $\vec{\gamma} \in (\Gamma')^n$. Then $\mathrm{tp}_{(\Omega',\mathbf{k}',\Gamma')}(t|\Omega)$ is determined by $\mathrm{tp}_{\mathbf{k}'}(\vec{a}|\mathbf{k})$, $\mathrm{tp}_{\Gamma'}(\vec{\gamma}|\Gamma)$ and the specification of a polynomial $P(X, Y, T) \in \Omega[X, Y, T]$ where $X = (X_1, \ldots, X_m)$, $Y = (Y_1, \ldots, Y_n)$ and T is a single indeterminate such that $P(\vec{a}, \vec{\gamma}, T) \in \Omega(\vec{a}, \vec{\gamma})[T]$ is irreducible and $P(\vec{a}, \vec{\gamma}, t) = 0$.

Finally, by the last argument of the proof of Lemma 7.7.1, all elements of Ω' outside $\Omega(\mathbf{k}' \cup \Gamma')^{\mathrm{ac}}$ realize the same type in $(\Omega', \mathbf{k}', \Gamma')$ over Ω.

Hence we have at most κ many different 1-types in $(\Omega', \mathbf{k}', \Gamma')$ over Ω. □

Remark. It is not hard to show that if $(\Omega, \mathbf{k}, \Gamma)$ is a model of T and $(\mathbf{k}, \Gamma)$ is a Mann pair, then the subsets $\mathbf{k}$ and Γ of Ω are definable in the structure $(\Omega, \mathbf{k} \cup \Gamma)$. Using this fact, Proposition 7.7.3 also follows from Fact 2.1 and Theorem 4.8 in [1].

7.7.3 Axiomatizing $(\Omega, \mathbf{k}, \Gamma)$

Our goal here is to show that if $(\Omega, \mathbf{k}, \Gamma)$ is a model of T and $(\mathbf{k}, \Gamma)$ is a Mann pair with $\mathbf{k}$ small in Ω, then the elementary theory of $(\Omega, \mathbf{k}, \Gamma)$ is completely determined by the elementary theories of the field $\mathbf{k}$ and of the group Γ. We achieve this after adding names for enough elements of $\mathbf{k}$ and Γ to witness that $(\mathbf{k}, \Gamma)$ is a Mann pair.

Fix a model $(\Omega_0, \mathbf{k}_0, \Gamma_0)$ of T such that $(\mathbf{k}_0, \Gamma_0)$ is a Mann pair. For each $n \geq 2$, fix a finite subset $\Gamma_0(n)$ of Γ_0^n such that

$$\Sigma_n^{\mathrm{nd}}(\mathbf{k}_0, \Gamma_0) = \bigcup_{\vec{\gamma} \in \Gamma_0(n)} \Sigma_n^{\mathrm{nd}}(\mathbf{k}_0, \Gamma_0; \vec{\gamma}) \times \Gamma_0 \vec{\gamma},$$

and for each $\vec{\gamma} \in \Gamma_0(n)$, fix a basis $B(\vec{\gamma}) \subseteq \mathbf{k}_0^n$ of the $\mathbf{k}_0$-linear subspace $\Sigma_n(\mathbf{k}_0, \Gamma_0; \vec{\gamma})$ of $\mathbf{k}_0^n$. Let $\mathcal{L}(\mathbf{k}_0, \Gamma_0)$ be the language $\mathcal{L}$ augmented by names for the elements of $\mathbf{k}_0 \cup \Gamma_0$.

In the following definition and subsequent remarks $(\Omega, \mathbf{k}, \Gamma)$ ranges over models of T that contain $(\Omega_0, \mathbf{k}_0, \Gamma_0)$ as a substructure. We construe such $(\Omega, \mathbf{k}, \Gamma)$ as an $\mathcal{L}(\mathbf{k}_0, \Gamma_0)$-structure in the obvious way, and say that $(\Omega, \mathbf{k}, \Gamma)$ *satisfies the Mann axioms of* $(\Omega_0, \mathbf{k}_0, \Gamma_0)$ if for each $n \geq 2$:

(1) $\Sigma_n^{\mathrm{nd}}(\mathbf{k}, \Gamma) = \bigcup_{\vec{\gamma} \in \Gamma_0(n)} \Sigma_n^{\mathrm{nd}}(\mathbf{k}, \Gamma; \vec{\gamma}) \times \Gamma \vec{\gamma}$;

(2) for each $\vec{\gamma} \in \Gamma_0(n)$, the $\mathbf{k}$-linear subspace $\Sigma_n(\mathbf{k}, \Gamma; \vec{\gamma})$ of $\mathbf{k}^n$ is generated by $B(\vec{\gamma})$;

Remarks. The reason for this terminology is that we have a set $\mathrm{Mann}(\mathbf{k}_0, \Gamma_0)$ of sentences in the language $\mathcal{L}(\mathbf{k}_0, \Gamma_0)$ such that for all $(\Omega, \mathbf{k}, \Gamma)$ as above,

$$(\Omega, \mathbf{k}, \Gamma) \models \mathrm{Mann}(\mathbf{k}_0, \Gamma_0)$$

$$\Longleftrightarrow$$

$$(\Omega, \mathbf{k}, \Gamma) \text{ satisfies the Mann axioms of } (\Omega_0, \mathbf{k}_0, \Gamma_0).$$

In particular, if $(\Omega, \mathbf{k}, \Gamma)$ is an elementary extension of $(\Omega_0, \mathbf{k}_0, \Gamma_0)$, then $(\Omega, \mathbf{k}, \Gamma)$ satisfies the Mann axioms of $(\Omega_0, \mathbf{k}_0, \Gamma_0)$.

Let $\mathcal{L}^{\mathrm{f}} := \{0, 1, -, +, \cdot\}$ be the one-sorted language whose symbols are those of $\mathcal{L}_{\Sigma}^{\mathrm{f,g}}$ involving only sort f, and let $\mathcal{L}^{\mathrm{f}}(\mathbf{k}_0)$ be $\mathcal{L}^{\mathrm{f}}$ augmented by names for the elements of $\mathbf{k}_0$. Let $\mathcal{L}^{\mathrm{g}} := \{1, {}^{-1}, \cdot\}$ be the one-sorted language whose symbols are those of $\mathcal{L}_{\Sigma}^{\mathrm{f,g}}$ involving only sort g, and let $\mathcal{L}^{\mathrm{g}}(\Gamma_0)$ be $\mathcal{L}^{\mathrm{g}}$ augmented by names for the elements of Γ_0. Let $n \geq 1$, let $x = (x_1, \dots, x_n)$ be a tuple of distinct f-variables, and let $y = (y_1, \dots, y_n)$ be a tuple of distinct g-variables. A careful look at the proof of Lemma 7.3.2 for $\vec{r} = (1, \dots, 1)$ yields the following: there are quantifier-free $\mathcal{L}^{\mathrm{f}}(\mathbf{k}_0)$-formulas $\phi_1(x), \dots, \phi_m(x)$ and quantifier-free $\mathcal{L}^{\mathrm{g}}(\Gamma_0)$-formulas $\psi_1(y), \dots, \psi_m(y)$ such that for all $(\Omega, \mathbf{k}, \Gamma) \models \mathrm{Mann}(\mathbf{k}_0, \Gamma_0)$,

$$\Sigma_n(\mathbf{k}^n, \Gamma^n) = \bigcup_{i=1}^{m} \phi_i(\mathbf{k}^n) \times \psi_i(\Gamma^n).$$

This uniformity is crucial in the proof of the next result.

Theorem 7.7.4. *Let* $(\Omega_1, \mathbf{k}_1, \Gamma_1)$ *and* $(\Omega_2, \mathbf{k}_2, \Gamma_2)$ *be models of* T *such that*

(1) $[\Omega_1 : \mathbf{k}_1] > 2$ *and* $[\Omega_2 : \mathbf{k}_2] > 2$*;*

(2) $(\Omega_0, \mathbf{k}_0, \Gamma_0) \subseteq (\Omega_1, \mathbf{k}_1, \Gamma_1)$ *and* $(\Omega_0, \mathbf{k}_0, \Gamma_0) \subseteq (\Omega_2, \mathbf{k}_2, \Gamma_2)$*;*

(3) $(\Omega_1, \mathbf{k}_1, \Gamma_1)$ *and* $(\Omega_2, \mathbf{k}_2, \Gamma_2)$ *satisfy the Mann axioms of* $(\Omega_0, \mathbf{k}_0, \Gamma_0)$*.*

Then: $\quad (\Omega_1, \mathbf{k}_1, \Gamma_1) \equiv_{\mathbf{k}_0 \cup \Gamma_0} (\Omega_2, \mathbf{k}_2, \Gamma_2) \iff \mathbf{k}_1 \equiv_{\mathbf{k}_0} \mathbf{k}_2$ *and* $\Gamma_1 \equiv_{\Gamma_0} \Gamma_2$.

Proof. The forward direction is clear. For the converse, assume $\mathbf{k}_1 \equiv_{\mathbf{k}_0} \mathbf{k}_2$ and $\Gamma_1 \equiv_{\Gamma_0} \Gamma_2$. Without loss we can also assume that $(\Omega_1, \mathbf{k}_1, \Gamma_1)$ and $(\Omega_2, \mathbf{k}_2, \Gamma_2)$ are κ-saturated, where $\kappa > |\mathbf{k}_0| + |\Gamma_0|$.

We construct a back-and-forth system between the $\mathcal{L}_{\Sigma}^{\mathrm{f,g}}$-structures $(\mathbf{k}_1, \Gamma_1)$ and $(\mathbf{k}_2, \Gamma_2)$. Let $\mathcal{S}_1$ be the set of substructures $(\mathbf{k}_1', \Gamma_1')$ of $(\mathbf{k}_1, \Gamma_1)$ that extend $(\mathbf{k}_0, \Gamma_0)$ and have cardinality $< \kappa$ (so $\mathbf{k}_1'$ is a subring, not necessarily a subfield, of $\mathbf{k}_1$, and Γ_1' is a subgroup of Γ_1). Define $\mathcal{S}_2$ likewise, with $(\mathbf{k}_2, \Gamma_2)$ in place of $(\mathbf{k}_1, \Gamma_1)$, and let $\mathcal{I}$ be the set of isomorphisms

$$\iota = (\iota_{\mathrm{f}}, \iota_{\mathrm{g}}) : (\mathbf{k}_1', \Gamma_1') \to (\mathbf{k}_2', \Gamma_2')$$

where $(\mathbf{k}_i', \Gamma_i') \in \mathcal{S}_i$ for $i = 1, 2$, such that ι is the identity on $(\mathbf{k}_0, \Gamma_0)$, $\iota_{\mathrm{f}} : \mathbf{k}_1' \to \mathbf{k}_2'$ is a partial elementary map from $\mathbf{k}_1$ to $\mathbf{k}_2$, and $\iota_{\mathrm{g}} : \Gamma_1' \to \Gamma_2'$ is a partial elementary map from Γ_1 to Γ_2. It is clear that the identity map of $(\mathbf{k}_0, \Gamma_0)$ belongs to $\mathcal{I}$.

Next we show that $\mathcal{I}$ is a back-and-forth system. By symmetry it is enough to show that we can go forth. Let

$$\iota = (\iota_{\mathrm{f}}, \iota_{\mathrm{g}}) : (\mathbf{k}_1', \Gamma_1') \to (\mathbf{k}_2', \Gamma_2')$$

be in $\mathcal{I}$. Replacing $\mathbf{k}_1'$ and $\mathbf{k}_2'$ by their fraction fields inside $\mathbf{k}_1$ and $\mathbf{k}_2$ and extending ι_{f} accordingly, without changing ι_{g}, we arrange that $\mathbf{k}_1'$ and $\mathbf{k}_2'$ are subfields of $\mathbf{k}_1$ and $\mathbf{k}_2$.

Let $a \in \mathbf{k}_1 \setminus \mathbf{k}_1'$ be given. Using saturation we can take $b \in \mathbf{k}_2 \setminus \mathbf{k}_2'$ and a field isomorphism $j : \mathbf{k}_1'(a) \to \mathbf{k}_2'(b)$ that extends ι_{f}, sends a to b, and is a partial elementary map from $\mathbf{k}_1$ to $\mathbf{k}_2$. It is obvious that $(\mathbf{k}_1'(a), \Gamma_1') \in \mathcal{S}_1$ and $(\mathbf{k}_2'(b), \Gamma_2') \in \mathcal{S}_2$, and it follows easily from the remark just before the lemma that $(j, \iota_{\mathrm{g}}) \in \mathcal{I}$.

Now let $\alpha \in \Gamma_1 \setminus \Gamma_1'$. By saturation we can find $\beta \in \Gamma_2 \setminus \Gamma_2'$ and a group isomorphism $h : \alpha^{\mathbb{Z}}\Gamma_1' \to \beta^{\mathbb{Z}}\Gamma_2'$ from Γ_1 to Γ_2 that extends ι_{g}, sends α to β, and is partial elementary map from Γ_1 to Γ_2. It is obvious that $(\mathbf{k}_1', \alpha^{\mathbb{Z}}\Gamma_1') \in \mathcal{S}_1$ and $(\mathbf{k}_2', \beta^{\mathbb{Z}}\Gamma_2') \in \mathcal{S}_2$. It follows easily from the remark just before the lemma that $(\iota_{\mathrm{f}}, h) : (\mathbf{k}_1', \alpha^{\mathbb{Z}}\Gamma_1') \to (\mathbf{k}_2', \beta^{\mathbb{Z}}\Gamma_2')$ belongs to $\mathcal{I}$.

We have now shown that $\mathcal{I}$ is a nonempty back-and-forth system. It follows that $(\mathbf{k}_1, \Gamma_1) \equiv_{(\mathbf{k}_0, \Gamma_0)} (\mathbf{k}_2, \Gamma_2)$. From (1), (3), and Lemma 7.7.2 we obtain that $\mathbf{k}_i \cup \Gamma_i$ is small in Ω_i for $i = 1, 2$. Then a proof like that of Lemma 7.7.1 yields

$$(\Omega_1, \mathbf{k}_1, \Gamma_1) \equiv_{\mathbf{k}_0 \cup \Gamma_0} (\Omega_2, \mathbf{k}_2, \Gamma_2) \iff \mathbf{k}_1 \equiv_{\mathbf{k}_0} \mathbf{k}_2 \text{ and } \Gamma_1 \equiv_{\Gamma_0} \Gamma_2,$$

as desired. □

For algebraically closed $\mathbf{k}_i$ this result takes the following form:

Corollary 7.7.5. *Let* $(\Omega_1, \mathbf{k}_1, \Gamma_1)$ *and* $(\Omega_2, \mathbf{k}_2, \Gamma_2)$ *be models of* T *such that*

(1) $\mathbf{k}_1$ *and* $\mathbf{k}_2$ *are algebraically closed,* $\mathbf{k}_1 \neq \Omega_1$, $\mathbf{k}_2 \neq \Omega_2$*;*

(2) $(\Omega_0, \mathbf{k}_0, \Gamma_0) \subseteq (\Omega_1, \mathbf{k}_1, \Gamma_1)$ *and* $(\Omega_0, \mathbf{k}_0, \Gamma_0) \subseteq (\Omega_2, \mathbf{k}_2, \Gamma_2)$*;*

(3) $(\Omega_1, \mathbf{k}_1, \Gamma_1)$ *and* $(\Omega_2, \mathbf{k}_2, \Gamma_2)$ *satisfy the Mann axioms of* $(\Omega_0, \mathbf{k}_0, \Gamma_0)$*.*

Then: $\quad (\Omega_1, \mathbf{k}_1, \Gamma_1) \equiv_{\mathbf{k}_0 \cup \Gamma_0} (\Omega_2, \mathbf{k}_2, \Gamma_2) \iff \Gamma_1 \equiv_{\Gamma_0} \Gamma_2.$

7.8 Definable sets in $(\Omega, \mathbf{k}, \Gamma)$

If $(\Omega, \mathbf{k}, \Gamma)$ is a model of T and $(\mathbf{k}, \Gamma)$ is a Mann pair, then Proposition 7.2 determines the subsets of $\mathbf{k}^m \times \Gamma^n$ that are definable in $(\Omega, \mathbf{k}, \Gamma)$, but in this section we wish to describe more generally the subsets of Ω^m definable in $(\Omega, \mathbf{k}, \Gamma)$. The main result of this kind is Theorem 7.8.5 below, where $\mathbf{k}$ is algebraically closed and Γ satisfies a mild extra condition.

A back-and-forth system. To define a back-and-forth system adequate for proving Theorem 7.8.5 is not so obvious, and we managed to do it only after much trial-and-error; see Lemmas 7.8.1, 7.8.2, 7.8.3. Throughout this subsection we fix a model $(\Omega_0, \mathbf{k}_0, \Gamma_0)$ of T such that $(\mathbf{k}_0, \Gamma_0)$ is a Mann pair.

Lemma 7.8.1. *Let* $(\Omega, \mathbf{k}, \Gamma)$ *be a model of* T *such that* $(\Omega_0, \mathbf{k}_0, \Gamma_0) \subseteq (\Omega, \mathbf{k}, \Gamma)$ *and* $(\Omega, \mathbf{k}, \Gamma)$ *satisfies the Mann axioms for* $(\Omega_0, \mathbf{k}_0, \Gamma_0)$*. Then*

(1) $\mathbf{k}$ *and* $\mathbf{k}_0(\Gamma)$ *are linearly disjoint over* $\mathbf{k}_0$*;*

(2) *if* $\mathbf{k}|\mathbf{k}_0$ *is regular and* $\Gamma|\Gamma_0$ *is pure, then* $\mathbf{k}(\Gamma)|\mathbf{k}_0(\Gamma_0)$ *is regular.*

Proof. Let $\gamma_1, \dots, \gamma_n \in \Gamma$ be linearly dependent over $\mathbf{k}$; it is enough to show that then they are linearly dependent over $\mathbf{k}_0$. We can arrange that $a_1\gamma_1 + \cdots + a_n\gamma_n = 0$ with $n \geq 2$ and nonzero $a_1, \dots, a_n \in \mathbf{k}$ such that $\sum_{i \in I} a_i\gamma_i \neq 0$ for all nonempty proper subsets I of $\{1, \dots, n\}$. Since $(\Omega, \mathbf{k}, \Gamma)$ satisfies the Mann axioms for $(\Omega_0, \mathbf{k}_0, \Gamma_0)$, this gives

$$(\gamma_1, \dots, \gamma_n) = \gamma\vec{\gamma}_0, \qquad \gamma \in \Gamma,\ \vec{\gamma}_0 \in \Gamma_0(n),$$

so $\gamma_1, \dots, \gamma_n$ are indeed linearly dependent over $\mathbf{k}_0$.

Suppose now that $\mathbf{k}|\mathbf{k}_0$ is regular and $\Gamma|\Gamma_0$ is pure. Then by (1) and Theorem 4.13 from [15] the extension $\mathbf{k}(\Gamma)|\mathbf{k}_0(\Gamma)$ is regular, and by Lemma 5.13 of [9] the extension $\mathbf{k}_0(\Gamma)|\mathbf{k}_0(\Gamma_0)$ is regular. Hence $\mathbf{k}(\Gamma)|\mathbf{k}_0(\Gamma_0)$ is regular, by Proposition 4.11(b) of Chapter VIII of [15]. □

Fix a cardinal $\kappa > |\Omega_0|$, and let $(\Omega, \mathbf{k}, \Gamma)$ be a κ-saturated model of T such that $(\Omega_0, \mathbf{k}_0, \Gamma_0) \subseteq (\Omega, \mathbf{k}, \Gamma)$ and $(\Omega, \mathbf{k}, \Gamma)$ satisfies the Mann axioms for $(\Omega_0, \mathbf{k}_0, \Gamma_0)$. Define $\mathrm{Sub}(\Omega, \mathbf{k}, \Gamma)$ to be the collection of all models $(\Omega', \mathbf{k}', \Gamma')$ of T such that:

(i) $\Omega_0 \subseteq \Omega' \subseteq \Omega$ (as fields) and $|\Omega'| < \kappa$;

(ii) $\mathbf{k}_0 \subseteq \mathbf{k}' \subseteq \mathbf{k}$ (as fields) and the extension $\mathbf{k}|\mathbf{k}'$ is regular;

(iii) $\Gamma_0 \subseteq \Gamma' \subseteq \Gamma$ (as groups), and the extension $\Gamma|\Gamma'$ is pure;

(iv) $\mathbf{k}(\Gamma)$ and Ω' are free over $\mathbf{k}'(\Gamma')$.

So $(\Omega', \mathbf{k}', \Gamma') \in \mathrm{Sub}(\Omega, \mathbf{k}, \Gamma)$ yields a diagram of field inclusions:

$$\begin{tikzcd}
 & & \Omega \\
 & \mathbf{k}(\Gamma) & \Omega' \\
\mathbf{k} & \mathbf{k}'(\Gamma') & \\
\mathbf{k}' & &
\end{tikzcd}$$

Lemma 7.8.2. *Let* $(\Omega', \mathbf{k}', \Gamma') \in \mathrm{Sub}(\Omega, \mathbf{k}, \Gamma)$*. Then*

(1) $\mathbf{k}(\Gamma)$ *and* Ω' *are linearly disjoint over* $\mathbf{k}'(\Gamma')$*,*

(2) $\mathbf{k}$ *and* Ω' *are linearly disjoint over* $\mathbf{k}'$*,*

(3) $\mathbf{k}(\Gamma') \cap \Gamma = \Gamma'$*,*

(4) $(\Omega', \mathbf{k}', \Gamma')$ *is a substructure of* $(\Omega, \mathbf{k}, \Gamma)$*.*

Proof. From the linear disjointness of $\mathbf{k}$ and $\mathbf{k}_0(\Gamma)$ over $\mathbf{k}_0$—item (1) of Lemma 7.8.1—we obtain the linear disjointness of $\mathbf{k}$ and $\mathbf{k}'(\Gamma)$ over $\mathbf{k}'$. Hence $\mathbf{k}(\Gamma)|\mathbf{k}'(\Gamma)$ is regular. We then argue as in the proof of (2) in Lemma 7.8.1, with $\mathbf{k}'$ instead of $\mathbf{k}_0$, that $\mathbf{k}'(\Gamma)|\mathbf{k}'(\Gamma')$ is regular, and so $\mathbf{k}(\Gamma)|\mathbf{k}'(\Gamma')$ is regular. Then (1) follows from condition (iv) above and Theorem 4.12 on page 367 of [15]. This also gives (2).

For (3), let $\gamma \in \mathbf{k}(\Gamma') \cap \Gamma$, so $\gamma = \frac{a_1\alpha_1 + \cdots + a_m\alpha_m}{b_1\beta_1 + \cdots + b_n\beta_n}$, with $a_1, \ldots, a_m, b_1, \ldots, b_n$ from $\mathbf{k}$, and $\alpha_1, \ldots, \alpha_m$, $\beta_1, \ldots, \beta_n$ from Γ', and $b_1\beta_1 + \cdots + b_n\beta_n \neq 0$. Taking such a representation of γ with minimal $m+n$, we have $m, n \geq 1$, and since $(\Omega, \mathbf{k}, \Gamma)$ satisfies the Mann axioms for $(\Omega_0, \mathbf{k}_0, \Gamma_0)$, this gives

$$(\alpha_1, \ldots, \alpha_m, \beta_1\gamma, \ldots, \beta_n\gamma) \in \Gamma\vec{\gamma}_0, \qquad \vec{\gamma}_0 \in \Gamma_0(m+n).$$

In particular, $\frac{\beta_1\gamma}{\alpha_1} \in \Gamma_0 \subseteq \Gamma'$, so $\gamma \in \Gamma'$.

As to (4), by (2) we have $\Omega' \cap \mathbf{k} = \mathbf{k}'$, so if $\gamma \in \Omega' \cap \Gamma$, then $\gamma \in \mathbf{k}'(\Gamma')$ by (1), and hence $\gamma \in \Gamma'$ by (3). $\square$

Next, for $i = 1, 2$, let $(\Omega_i, \mathbf{k}_i, \Gamma_i)$ be a κ-saturated model of T such that $(\Omega_0, \mathbf{k}_0, \Gamma_0) \subseteq (\Omega_i, \mathbf{k}_i, \Gamma_i)$, $[\Omega_i : \mathbf{k}_i] > 2$, and $(\Omega_i, \mathbf{k}_i, \Gamma_i)$ satisfies the Mann axioms of $(\Omega_0, \mathbf{k}_0, \Gamma_0)$, and put $\mathrm{Sub}_i := \mathrm{Sub}(\Omega_i, \mathbf{k}_i, \Gamma_i)$.

Let $\mathcal{I}$ be the set of isomorphisms

$$\iota : (\Omega_1', \mathbf{k}_1', \Gamma_1') \to (\Omega_2', \mathbf{k}_2', \Gamma_2'), \qquad (\Omega_i', \mathbf{k}_i', \Gamma_i') \in \mathrm{Sub}_i \text{ for } i = 1, 2,$$

that are the identity on $\mathbf{k}_0$ and on Γ_0, such that $\iota|\mathbf{k}_1'$ is a partial elementary map from $\mathbf{k}_1$ to $\mathbf{k}_2$, and $\iota|\Gamma_1'$ is a partial elementary map from Γ_1 to Γ_2; we do *not* require that ι is the identity on Ω_0.

Lemma 7.8.3. *$\mathcal{I}$ is a (possibly empty) back-and-forth system.*

Proof. Let $\iota : (\Omega_1', \mathbf{k}_1', \Gamma_1') \to (\Omega_2', \mathbf{k}_2', \Gamma_2')$ be in $\mathcal{I}$, and $r \in \Omega_1 \setminus \Omega_1'$; by symmetry it is enough to show that then ι extends to an isomorphism in $\mathcal{I}$ that has r in its domain.

First, consider the case that $r \in \mathbf{k}_1$. Then we take a field $\mathbf{k}_1'' \preceq \mathbf{k}_1$ such that $\mathbf{k}_1'(r) \subseteq \mathbf{k}_1''$ and $|\mathbf{k}_1''| < \kappa$. Using saturation we extend $\iota|\mathbf{k}_1'$ to a field isomorphism $f : \mathbf{k}_1'' \to \mathbf{k}_2'' \preceq \mathbf{k}_2$ that is a partial elementary map between $\mathbf{k}_1$ and $\mathbf{k}_2$. It is clear that $\mathbf{k}_i|\mathbf{k}_i''$ is regular, and that $\mathbf{k}_i(\Gamma_i)$ and Ω_i'' are free over $\mathbf{k}_i''(\Gamma_i')$, where $\Omega_i'' := (\Omega_i'\mathbf{k}_i'')^{\mathrm{ac}}$ and $i \in \{1, 2\}$. So $(\Omega_i'', \mathbf{k}_i'', \Gamma_i') \in \mathrm{Sub}_i$ for $i = 1, 2$. Then by (2) of Lemma 7.8.2 we have a common extension of ι and f to an isomorphism

$$(\Omega_1'', \mathbf{k}_1'', \Gamma_1') \to (\Omega_2'', \mathbf{k}_2'', \Gamma_2')$$

in $\mathcal{I}$; it has r in its domain.

Next, assume that $r \in \Gamma_1$. Then we take a group $\Gamma_1'' \preceq \Gamma_1$ such that $r^{\mathbb{Z}}\Gamma_1' \subseteq \Gamma_1''$ and $|\Gamma_1''| < \kappa$. Using saturation we extend $\iota|\Gamma_1'$ to a group isomorphism $g : \Gamma_1'' \to \Gamma_2'' \preceq \Gamma_2$ that is a partial elementary map between Γ_1 and Γ_2. It is clear that $\Gamma_i|\Gamma_i''$ is pure, and that $\mathbf{k}_i(\Gamma_i)$ and Ω_i'' are free over $\mathbf{k}_i'(\Gamma_i'')$, where $\Omega_i'' := \big(\Omega_i'(\Gamma_i'')\big)^{\mathrm{ac}}$ and $i \in \{1, 2\}$. So $(\Omega_i'', \mathbf{k}_i', \Gamma_i'') \in \mathrm{Sub}_i$ for $i = 1, 2$. Also, $(\Omega_i, \mathbf{k}_i, \Gamma_i)$ satisfies the Mann axioms of $(\Omega_0, \mathbf{k}_0, \Gamma_0)$, for $i = 1, 2$, and thus we have a field isomorphism $h : \mathbf{k}_1'(\Gamma_1'') \to \mathbf{k}_2'(\Gamma_2'')$ that extends $\iota|\mathbf{k}_1'$ and g. Then by (1) of Lemma 7.8.2 this gives a common extension of ι and h to an isomorphism

$$(\Omega_1'', \mathbf{k}_1', \Gamma_1'') \to (\Omega_2'', \mathbf{k}_2', \Gamma_2'')$$

in $\mathcal{I}$; it has r in its domain.

If $r \in \Omega_1'(\mathbf{k}_1 \cup \Gamma_1)^{\mathrm{ac}}$, then we can take a finite number of steps of the two types above to extend ι to an element of $\mathcal{I}$ with r in its domain.

Finally, suppose that $r \notin \Omega_1'(\mathbf{k}_1 \cup \Gamma_1)^{\mathrm{ac}}$. By saturation and the smallness assumption $[\Omega_2 : \mathbf{k}_2] > 2$ we can take $s \in \Omega_2$ with $s \notin \Omega_2'(\mathbf{k}_2 \cup \Gamma_2)^{\mathrm{ac}}$. With

$$\Omega_1'' := \Omega_1'(r)^{\mathrm{ac}}, \qquad \Omega_2'' := \Omega_2'(s)^{\mathrm{ac}},$$

it is clear that $(\Omega_i'', \mathbf{k}_i', \Gamma_i') \in \mathrm{Sub}_i$ for $i = 1, 2$. We can extend ι to a field isomorphism $\Omega_1'' \to \Omega_2''$ that sends r to s, and this gives an isomorphism

$$(\Omega_1'', \mathbf{k}_1', \Gamma_2') \to (\Omega_2'', \mathbf{k}_2', \Gamma_2')$$

in $\mathcal{I}$ with r in its domain. □

Corollary 7.8.4. *Suppose that* $(\Omega_0, \mathbf{k}_0, \Gamma_0) \subseteq (\Omega, \mathbf{k}, \Gamma) \models T$ *and* $[\Omega_0 : \mathbf{k}_0] > 2$. *Then* $(\Omega_0, \mathbf{k}_0, \Gamma_0) \preceq (\Omega, \mathbf{k}, \Gamma)$ *if and only if* $(1) - (4)$ *below are satisfied:*

(1) $[\Omega : \mathbf{k}] > 2$*;*

(2) $(\Omega, \mathbf{k}, \Gamma)$ *satisfies the Mann axioms for* $(\Omega_0, \mathbf{k}_0, \Gamma_0)$*;*

(3) $\mathbf{k}_0 \preceq \mathbf{k}$ *and* $\Gamma_0 \preceq \Gamma$*;*

(4) $\mathbf{k}(\Gamma)$ *and* Ω_0 *are free over* $\mathbf{k}_0(\Gamma_0)$*.*

Proof. It is easy to check that if $(\Omega_0, \mathbf{k}_0, \Gamma_0) \preceq (\Omega, \mathbf{k}, \Gamma)$, then (1)–(4) hold. For the converse, assume (1)-(4). By passing to an elementary extension of $(\Omega, \mathbf{k}, \Gamma)$ we arrange that $(\Omega, \mathbf{k}, \Gamma)$ is κ-saturated. Put $(\Omega_1, \mathbf{k}_1, \Gamma_1) := (\Omega, \mathbf{k}, \Gamma)$; let $(\Omega_2, \mathbf{k}_2, \Gamma_2)$ be a κ-saturated elementary extension of $(\Omega_0, \mathbf{k}_0, \Gamma_0)$. Then $(\Omega_0, \mathbf{k}_0, \Gamma_0) \in \mathrm{Sub}(\Omega_i, \mathbf{k}_i, \Gamma_i)$ for $i = 1, 2$. Let $\mathcal{I}$ be the back-and-forth system considered in the previous lemma. Then the identity map on $(\Omega_0, \mathbf{k}_0, \Gamma_0)$ belongs to $\mathcal{I}$, and so $(\Omega_1, \mathbf{k}_1, \Gamma_1)$ and $(\Omega_2, \mathbf{k}_2, \Gamma_2)$ are elementarily equivalent over Ω_0. Thus $(\Omega_0, \mathbf{k}_0, \Gamma_0) \preceq (\Omega, \mathbf{k}, \Gamma)$. □

Definable Sets. We now specify the two unary relation symbols of $\mathcal{L}$ to be U and V, to be interpreted in a model of T as the underlying set of the distinguished subfield and of the distinguished multiplicative group, respectively. Let a model $(\Omega, \mathbf{k}, \Gamma)$ of T be given and let $x = (x_1, \dots, x_m)$ be a tuple of distinct variables. Call a subset of Ω^m *special* if it is defined in $(\Omega, \mathbf{k}, \Gamma)$ by a special formula in $x = (x_1, \dots, x_m)$, that is, a formula

$$\exists y \; \exists z (U(y) \wedge V(z) \wedge \phi(x, y, z)),$$

where $x_1, \ldots, x_m, y_1, \ldots, y_s, z_1, \ldots, z_t$ are distinct variables, $y = (y_1, \ldots, y_s)$, $z = (z_1, \ldots, z_t)$ and $\phi(x, y, z)$ is a quantifier-free formula in the language of rings augmented by names for the elements of Ω, and where $U(y)$ and $V(z)$ abbreviate $U(y_1) \wedge \cdots \wedge U(y_s)$ and $V(z_1) \wedge \cdots \wedge V(z_t)$, respectively.

Theorem 7.8.5. *Let $(\Omega, \mathbf{k}, \Gamma)$ be a model of T such that $\mathbf{k}$ is an algebraically closed field, $\mathbf{k} \neq \Omega$, $(\mathbf{k}, \Gamma)$ is a Mann pair, and $\Gamma/\Gamma^{[p]}$ is finite for each p. Then the subsets of Ω^m definable in $(\Omega, \mathbf{k}, \Gamma)$ are exactly the boolean combinations in Ω^m of special subsets of Ω^m.*

Proof. We take $\kappa := \aleph_1$, and may assume that $(\Omega, \mathbf{k}, \Gamma)$ is κ-saturated. Let $(\Omega_0', \mathbf{k}_0, \Gamma_0)$ be a countable elementary substructure of $(\Omega, \mathbf{k}, \Gamma)$, and let Ω_0 be the algebraic closure of $\mathbf{k}_0(\Gamma_0)$ in Ω_0'. Then $(\Omega_0, \mathbf{k}_0, \Gamma_0) \subseteq (\Omega, \mathbf{k}, \Gamma)$, $(\mathbf{k}_0, \Gamma_0)$ is a Mann pair, and $(\Omega, \mathbf{k}, \Gamma)$ satisfies the Mann axioms of $(\Omega_0, \mathbf{k}_0, \Gamma_0)$. Let $\vec{r} = (r_1, \ldots, r_m)$ and $\vec{s} = (s_1, \ldots, s_m)$ be two tuples from Ω^m that satisfy the same special formulas in x using only names for elements of $A := \mathbf{k}_0 \cup \Gamma_0$; it suffices to show that then they realize the same type in $(\Omega, \mathbf{k}, \Gamma)$ over A.

For $i = 1, 2$, put $(\Omega_i, \mathbf{k}_i, \Gamma_i) := (\Omega, \mathbf{k}, \Gamma)$, so the structures in $\mathrm{Sub}(\Omega_i, \mathbf{k}_i, \Gamma_i)$ are all countable. Let $\mathcal{I}$ be the back-and-forth system of Lemma 7.8.3; it is enough to construct an isomorphism in this system that takes $\vec{r}$ to $\vec{s}$.

Let d be the transcendence degree of $\mathbf{k}(\Gamma)(\vec{r})$ over $\mathbf{k}(\Gamma)$. We can assume that $r_1, \ldots, r_d$ are algebraically independent over $\mathbf{k}(\Gamma)$. As in the proof of Theorem 3.8 in [9] it follows that $s_1, \ldots, s_d$ are algebraically independent over $\mathbf{k}(\Gamma)$, and $s_{d+1}, \ldots, s_m$ are algebraic over $\mathbf{k}(\Gamma)(s_1, \ldots, s_d)$.

Take some countable $(\Omega', \mathbf{k}', \Gamma') \preceq (\Omega, \mathbf{k}, \Gamma)$ such that $\Omega' \supseteq \Omega_0$ and $\mathbf{k}'(\Gamma')(\vec{r})$ has transcendence degree d over $\mathbf{k}'(\Gamma')$. In particular, $(\Omega', \mathbf{k}', \Gamma') \in \mathrm{Sub}(\Omega, \mathbf{k}, \Gamma)$. Let $a = (a_0, a_1, a_2, \ldots)$ be an enumeration of $\mathbf{k}'$, let $g = (g_0, g_1, g_2, \ldots)$ be an enumeration of Γ', and let $y_0, y_1, y_2, \ldots, z_0, z_1, z_2, \ldots$ be distinct variables, also distinct from $x_1, \ldots, x_m$, and put

$$y = (y_0, y_1, y_2, \ldots), \qquad z = (z_0, z_1, z_2, \ldots,).$$

Suppose $\psi_1(y), \ldots, \psi_k(y)$ are quantifier-free formulas in the language of rings augmented by names for the elements of $\mathbf{k}_0$, and $\theta_1(z), \ldots, \theta_k(z)$ are quantifier-free formulas in the language of groups augmented by names for the elements of Γ_0, and $\phi_1(x, y, z), \ldots, \phi_k(x, y, z)$ are quantifier-free formulas in the language of rings augmented by names for the elements of A, such that $\mathbf{k} \models \psi_j(a)$, $\Gamma \models \theta_j(g)$ and $(\Omega, \mathbf{k}, \Gamma) \models \phi_j(\vec{r}, a, g)$ for $j = 1, \ldots, k$. Then

$$(\Omega, \mathbf{k}, \Gamma) \models \exists y \exists z (U(y) \wedge V(z) \wedge \psi(y) \wedge \theta(z) \wedge \phi(\vec{r}, y, z)), \qquad \text{where}$$

$$\psi(y) := \bigwedge_j \psi_j(y), \quad \theta(z) := \bigwedge_j \theta_j(z), \quad \varphi(x, y, z) := \bigwedge_j \varphi_j(x, y, z).$$

The assumption on $\vec{r}$ and $\vec{s}$ then gives

$$(\Omega, \mathbf{k}, \Gamma) \models \exists y \exists z (U(y) \wedge V(z) \wedge \psi(y) \wedge \theta(z) \wedge \phi(\vec{s}, y, z)).$$

Hence we have a partial y, z-type over $A\vec{s}$ in $(\Omega, \mathbf{k}, \Gamma)$ consisting of the formulas $U(y_i)$ and $V(z_i)$ for $i = 0, 1, 2, \ldots$, the quantifier-free formulas $\psi(y)$ in the language of rings augmented by names for the elements of $\mathbf{k}_0$ such that $\mathbf{k} \models \psi(a)$, the quantifier-free formulas $\theta(z)$ in the language of groups augmented by names for the elements of Γ_0 such that $\Gamma \models \theta(g)$, and the formulas $\phi(\vec{s}, y, z)$ such that $\phi(x, y, z)$ is a quantifier-free formula in the language of rings augmented by names for the elements of A and $(\Omega, \mathbf{k}, \Gamma) \models \phi(\vec{r}, a, g)$. Let b, h with $b = (b_0, b_1, b_2, \ldots) \in \mathbf{k}^{\mathbb{N}}$ and $h = (h_0, h_1, h_2, \ldots) \in \Gamma^{\mathbb{N}}$ realize this y, z-type in $(\Omega, \mathbf{k}, \Gamma)$. Then $\{b_0, b_1, b_2, \ldots\}$ is the underlying set of a field $\mathbf{k}'' \preceq \mathbf{k}$ and we have a field isomorphism

$$\iota^{\mathrm{f}} : \mathbf{k}' \to \mathbf{k}'', \qquad \iota^{\mathrm{f}}(a_n) = b_n \text{ for all } n.$$

Likewise, $\{h_0, h_1, h_2, \ldots\}$ is the underlying set of a group $\Gamma'' \preceq \Gamma$ and we have a group isomorphism

$$\iota^{\mathrm{g}} : \Gamma' \to \Gamma'', \qquad \iota^{\mathrm{g}}(g_n) = h_n \text{ for all } n.$$

Note that ι^{f} is a partial elementary map from $\mathbf{k}$ to itself and is the identity on $\mathbf{k}_0$. Likewise, ι^{g} is a partial elementary map from Γ to itself and is the identity on Γ_0. Moreover, ι^{f} and ι^{g} have a common extension to a field isomorphism

$$\iota : \mathbf{k}'(\Gamma')(\vec{r}) \cong \mathbf{k}''(\Gamma'')(\vec{s})$$

sending $\vec{r}$ to $\vec{s}$. Put $\Omega_1' := \mathbf{k}'(\Gamma')(\vec{r})^{\mathrm{ac}}$ and $\Omega_2' := \mathbf{k}''(\Gamma'')(\vec{s})^{\mathrm{ac}}$. Then $(\Omega_1', \mathbf{k}', \Gamma'), (\Omega_2', \mathbf{k}'', \Gamma'') \in \mathrm{Sub}(\Omega, \mathbf{k}, \Gamma)$, and we have an isomorphism

$$(\Omega_1', \mathbf{k}', \Gamma') \cong (\Omega_2', \mathbf{k}'', \Gamma'')$$

that extends ι. It carries $\vec{r}$ to $\vec{s}$ and belongs to $\mathcal{I}$. $\square$

Note that if Γ is divisible or of finite rank, then the condition in the theorem that $\Gamma/\Gamma^{[p]}$ is finite for each p is satisfied.

Chapter 8

The real field with a subgroup of the unit circle

8.1 Oriented abelian groups

In this section, we use additive notation for abelian groups, unless specified otherwise. Let G be an ordered abelian group with distinguished element $1 > 0$. We define an abelian group $G_{\mathrm{mod}1}$ as follows: the underlying set of $G_{\mathrm{mod}1}$ is the subset

$$[0,1) := \{g \in G :\ 0 \le g < 1\}$$

of G, and its addition operation $+_1$ is "addition modulo 1", that is, for $g, h \in [0,1)$,

$$g +_1 h = g + h \ \text{ if } g + h < 1, \quad g +_1 h = g + h - 1 \ \text{ otherwise.}$$

Let $G(1)$ be the convex hull of $\mathbb{Z} \cdot 1$ in G. For $x \in G(1)$ we define $x_{\mathrm{mod}1} \in [0,1)$ by $x - x_{\mathrm{mod}1} \in \mathbb{Z} \cdot 1$, so we have a surjective group morphism

$$x \mapsto x_{\mathrm{mod}1} :\ G(1) \to G_{\mathrm{mod}1}$$

with kernel $\mathbb{Z} \cdot 1$. We equip $G_{\mathrm{mod}1}$ with the ternary relation $\mathcal{O}$ on its underlying set defined as follows: for $g, h, k \in [0,1)$,

$$\mathcal{O}(g,h,k) \ :\Longleftrightarrow\ g < h < k, \ \text{or } h < k < g, \ \text{or } k < g < h.$$

It is easy to check that for all $g, h, k \in [0,1)$ we have

$$\mathcal{O}(g,h,k) \iff \text{there are } x, y, z \in G(1) \text{ such that } x < y < z,\ z - x < 1,$$
$$x_{\mathrm{mod}1} = g,\ y_{\mathrm{mod}1} = h,\ z_{\mathrm{mod}1} = k.$$

The relation $\mathcal{O}$ is an orientation on $G_{\mathrm{mod}1}$ in the following sense: Let A be an abelian group. An *orientation* on A is a ternary relation $\mathcal{O}$ on A such that for all $a, b, c, d \in A$:

(1) $\{(x,y) \in A^2 :\ \mathcal{O}(0,x,y)\}$ is a strict linear order on the set $A \setminus \{0\}$,

(2) $\mathcal{O}(a,b,c) \ \Rightarrow \ \mathcal{O}(b,c,a)$,

(3) $\mathcal{O}(a,b,c) \ \Rightarrow \ \mathcal{O}(a+d,b+d,c+d)$,

Here it is part of (1) that if $x,y \in A$ and $\mathcal{O}(0,x,y)$, then $x \neq 0$ and $y \neq 0$.

Example. Let $\mathbb{S} := \{(a,b) \in \mathbb{R}^2 : a^2+b^2=1\}$ be the (multiplicative) circle group. It has identity $(1,0)$ and multiplication given by

$$(a_1,b_1)\cdot(a_2,b_2) = (a_1a_2 - b_1b_2, a_1b_2 + a_2b_1).$$

We have a group isomorphism

$$t \mapsto (\cos 2\pi t, \sin 2\pi t) \ : \ \mathbb{R}_{\mathrm{mod}1} \to \mathbb{S},$$

and we give $\mathbb{S}$ the orientation that makes this an isomorphism of oriented groups. It is tedious but not hard to construct a quantifier-free formula $\mathcal{O}(x_1,y_1,x_2,y_2,x_3,y_3)$ in the sublanguage $\{0,<\}$ of the language of ordered rings such that

$$\mathcal{O}\big((a_1,b_1),(a_2,b_2),(a_3,b_3)\big) \iff \mathbb{R} \models \mathcal{O}(a_1,b_1,a_2,b_2,a_3,b_3).$$

for all $(a_1,b_1),(a_2,b_2),(a_3,b_3) \in \mathbb{S}$. Thus $\mathbb{R}_{\mathrm{mod}1}$ and $\mathbb{S}$ are definable groups in the field $\mathbb{R}$ of real numbers, and are isomorphic as groups. Note that the isomorphism indicated above is not definable in the ordered field $\mathbb{R}$, but is definable in this ordered field expanded by the restriction of the sine function to $[0,2\pi]$.

For each real closed field R we put

$$\mathbb{S}(R) := \{(a,b) \in R^2 : a^2+b^2=1\}$$

and consider $\mathbb{S}(R)$ as a commutative group with identity $(1,0)$ and multiplication given by the same identity used to define the multiplication of $\mathbb{S}$. We also give it the orientation defined by the equivalence above involving the formula $\mathcal{O}(x_1,y_1,x_2,y_2,x_3,y_3)$, but with R instead of $\mathbb{R}$, and $\mathbb{S}(R)$ instead of $\mathbb{S}$.

An *oriented abelian group* is an abelian group with an orientation on it. We are going to show that every oriented abelian group is isomorphic to $G_{\mathrm{mod}1}$ for some ordered abelian group G with distinguished element $1>0$. In the rest of this section A is an oriented abelian group with orientation $\mathcal{O}$, and we let a,b,c,d range over A.

We first collect some facts about oriented abelian groups.

Observations.

(i) Given any a, the set $\{(x,y) \in A^2 : \mathcal{O}(a,x,y)\}$ is a strict linear order on $A \setminus \{a\}$, to be denoted by $<_a$, so $b <_a c$ means $\mathcal{O}(a,b,c)$, and $b <_a c$ implies in particular that a, b, c are distinct.

(ii) $a <_0 b \;\Rightarrow\; (-b <_0 a - b \;\&\; b - a <_0 -a)$.

(iii) $\mathcal{O}(0,a,b) \;\Leftrightarrow\; \mathcal{O}(0,-b,-a)$.

(iv) $\mathcal{O}(a,b,c) \;\Leftrightarrow\; \mathcal{O}(-c,-b,-a)$.

(v) $\big(\mathcal{O}(a,b,c) \;\&\; \mathcal{O}(0,a,c)\big) \;\Rightarrow \mathcal{O}(0,a,b)$.

(vi) $\big(\mathcal{O}(a,b,d) \;\&\; \mathcal{O}(b,c,d)\big) \;\Rightarrow\; \mathcal{O}(a,b,c)$.

It is easy to obtain (i) and (ii) from the axioms. Assume that (iii) does not hold and take a, b such that $\mathcal{O}(0,a,b)$ and $\mathcal{O}(0,-a,-b)$. We get $b - a <_0 -a$ and $-b <_0 a - b$ from (ii). Then using transitivity of $<_0$ twice we conclude that $b - a <_0 a - b$. By symmetric arguments we can get $a - b <_0 b - a$, which contradicts the assumption that $<_0$ is a strict linear order. Now (iv) easily follows from (iii). To see why (v) holds, assume $\mathcal{O}(a,b,c)$ and $\mathcal{O}(0,a,c)$. Then $\mathcal{O}(a,c,0)$, so $b <_a c$ and $c <_a 0$, hence $b <_a 0$, so $\mathcal{O}(a,b,0)$, and thus $\mathcal{O}(0,a,b)$. To get (vi) we first state (v) as the implication

$$(\mathcal{O}(b,c,d) \;\&\; \mathcal{O}(0,b,d)) \Rightarrow \mathcal{O}(0,b,c).$$

By axiom (3) we can replace 0 here by any element of A, and this gives (vi).

Originally we had (iii) as an axiom for orientations, but Jonathan Kirby pointed out how it followed from the other axioms.

Lemma 8.1.1. *We have the following equivalences:*

(1) $(a <_0 -b \text{ and } a + b <_0 -c) \iff (b <_0 -c \text{ and } a <_0 -b - c)$,

(2) $(-b <_0 a \text{ and } -c <_0 a + b) \iff (-c <_0 b \text{ and } -a <_0 b + c)$.

Proof. We only show (1), since (2) can be proved in a similar way. If the forward direction of (1) holds, then the other direction follows by switching a and c. So assume $a <_0 -b$ and $a + b <_0 -c$; we want to show $b <_0 -c$ and $a <_0 -b - c$. By observation (ii) above we get $b <_0 a + b$, hence $b <_0 -c$.

From $\mathcal{O}(0, a + b, -c)$ we get $\mathcal{O}(-b, a, -b - c)$ by adding $-b$, so $\mathcal{O}(a, -b - c, -b)$. In combination with $\mathcal{O}(0, a, -b)$ this gives $a <_0 -b - c$ by observation (v). □

Consider an ordered abelian group B with distinguished element $1 > 0$, and let $a, b \in [0,1) \subseteq B$. Then we have the following equivalences:

$$\begin{aligned}
a + b < 1 &\;\Leftrightarrow\; a = 0 \;\text{ or } b = 0 \;\text{ or } a <_0 -b,\\
a + b = 1 &\;\Leftrightarrow\; a \neq 0 \;\text{ and } b \neq 0 \;\text{ and } a +_1 b = 0,\\
a + b > 1 &\;\Leftrightarrow\; a \neq 0 \;\text{ and } b \neq 0 \;\text{ and } -b <_0 a.
\end{aligned}$$

Here the lefthand sides use addition in B and the righthand sides refer to the operations and relations on $B_{\text{mod}1}$, in particular, $-b$ as used on the righthand sides denotes the negative of b in $B_{\text{mod}1}$. This motivates the following case distinctions in the abstract setting of any oriented abelian group A:

$$\begin{aligned}
(a,b) \prec 1 &\;:\Leftrightarrow\; a = 0 \;\text{ or } b = 0 \;\text{ or } a <_0 -b,\\
(a,b) \asymp 1 &\;:\Leftrightarrow\; a \neq 0 \;\text{ and } b \neq 0 \;\text{ and } a + b = 0,\\
(a,b) \succ 1 &\;:\Leftrightarrow\; a \neq 0 \;\text{ and } b \neq 0 \;\text{ and } -b <_0 a.
\end{aligned}$$

Let a, b be given. Then exactly one of $(a,b) \prec 1$, $(a,b) \asymp 1$, $(a,b) \succ 1$ holds, and $(a,b) \prec 1$ iff $(b,a) \prec 1$, $(a,b) \asymp 1$ iff $(b,a) \asymp 1$, and $(a,b) \succ 1$ iff $(b,a) \succ 1$. We also define $(a,b) \preceq 1$ to mean that $(a,b) \prec 1$ or $(a,b) \asymp 1$, and define $(a,b) \succeq 1$ to mean that $(a,b) \succ 1$ or $(a,b) \asymp 1$.

To construct an ordered abelian group G with an element $1 > 0$ such that A and $G_{\text{mod}1}$ are isomorphic as oriented abelian groups, we set $A^* = \mathbb{Z} \times A$, and define the binary operation $+$ on A^* as follows:

$$(k,a) + (l,b) = \begin{cases} (k+l, a+b) & \text{if } (a,b) \prec 1,\\ (k+l+1, a+b) & \text{otherwise,} \end{cases}$$

where $k, l \in \mathbb{Z}$. We shall prove that A^* is an abelian group with $+$ as its addition operation. It is clear that $(0,0) + (k,a) = (k,a)$, $(k,a) + (-k,-a) = (0,0)$ if $(a,-a) \prec 1$, and $(k,a) + (-k-1,-a) = (0,0)$ if $(a,-a) \not\prec 1$. It is also clear that the operation $+$ on A^* is commutative. Verifying its associativity requires many case distinctions. To help in this we define:

$$((a,b),c) \prec 1 \;:\Leftrightarrow\; (a,b) \prec 1 \text{ and } (a+b,c) \prec 1$$

$$\begin{aligned}
1 \preceq ((a,b),c) \prec 2 \;&:\Leftrightarrow \big((a,b) \prec 1 \text{ and } (a+b,c) \succeq 1)\big) \\
&\qquad \text{or } \big((a,b) \succeq 1 \text{ and } (a+b,c) \prec 1\big) \\
2 \preceq ((a,b),c) \;&:\Leftrightarrow (a,b) \succeq 1 \text{ and } (a+b,c) \succeq 1.
\end{aligned}$$

Likewise, we define

$$\begin{aligned}
(a,(b,c)) \prec 1 \;&:\Leftrightarrow (b,c) \prec 1 \text{ and } (a,b+c) \prec 1 \\
1 \preceq (a,(b,c)) \prec 2 \;&:\Leftrightarrow \big((b,c) \prec 1 \text{ and } (a,b+c) \succeq 1)\big) \\
&\qquad \text{or } \big((b,c) \succeq 1 \text{ and } (a,b+c) \prec 1\big) \\
2 \preceq (a,(b,c)) \;&:\Leftrightarrow (b,c) \succeq 1 \text{ and } (a,b+c) \succeq 1.
\end{aligned}$$

Given a,b,c, exactly one of the three statements $((a,b),c) \prec 1$, $1 \preceq ((a,b),c) \prec 2$, $2 \preceq ((a,b),c)$ holds, and also exactly one of the three statements $(a,(b,c)) \prec 1$, $1 \preceq (a,(b,c)) \prec 2$, $2 \preceq (a,(b,c))$ holds.

Let $j,k,l \in \mathbb{Z}$; using the above notation, we have

$$((j,a)+(k,b))+(l,c) = \begin{cases} (j+k+l, a+b+c) & \text{if } ((a,b),c) \prec 1 \\ (j+k+l+1, a+b+c) & \text{if } 1 \preceq ((a,b),c) \prec 2 \\ (j+k+l+2, a+b+c) & \text{if } 2 \preceq ((a,b),c) \end{cases}$$

and likewise,

$$(j,a)+((k,b))+(l,c)) = \begin{cases} (j+k+l, a+b+c) & \text{if } (a,(b,c)) \prec 1 \\ (j+k+l+1, a+b+c) & \text{if } 1 \preceq (a,(b,c)) \prec 2 \\ (j+k+l+2, a+b+c) & \text{if } 2 \preceq (a,(b,c)) \end{cases}$$

So checking associativity of $+$ reduces to verifying the equivalences

$$\begin{aligned}
&((a,b),c) \prec 1 \;\Leftrightarrow\; (a,(b,c)) \prec 1, \quad 1 \preceq ((a,b),c) \prec 2 \Leftrightarrow\; 1 \preceq (a,(b,c)) \prec 2, \\
&2 \preceq ((a,b),c) \Leftrightarrow\; 2 \preceq (a,(b,c)).
\end{aligned}$$

The first equivalence follows by making the obvious case distinctions, with the main case handled by (1) of Lemma 8.1.1. The third equivalence follows likewise by (2) of Lemma 8.1.1, and the second equivalence follows from the first and third.

We define a strict linear order on the set A^* by

$$(k,a) < (l,b) \;:\Leftrightarrow\; k < l \text{ or } (k = l \text{ and } a <_0 b) \text{ or } (k = l \text{ and } a = 0 \text{ and } b \neq 0).$$

Let $j, k, l \in \mathbb{Z}$, and assume that $(j,a) < (k,b)$. We claim that then

$$(j,a) + (l,c) < (k,b) + (l,c).$$

If $j + 1 < k$, then $j + l + 1 < k + l$, and so our claim holds. Other cases split into various subcases that can be handled using the observations(i)–(vi). We only indicate the main points to be verified.

Suppose $j + 1 = k$. Then a tedious checking of cases shows that the claim holds.

Suppose $j = k$ and $a <_0 b$. If $(b,c) \prec 1$, then $(a,c) \prec 1$, so $(j,a) + (l,c) = (j + l, a + c)$, $(k,b) + (l,c) = (k + l, b + c)$, and a tedious checking of cases gives $\mathcal{O}(0, a + c, b + c)$, and our claim holds. If $(a,c) \prec 1$ and $(b,c) \succeq 1$, then $(j,a) + (l,c) = (j + l, a + c)$, $(k,b) + (l,c) = (k + l + 1, b + c)$, so the claim holds. Finally, if $(a,c) \succeq 1$ and $(b,c) \succeq 1$, then $(j,a) + (l,c) = (j + l + 1, a + c)$, $(k,b) + (l,c) = (k + l + 1, b + c)$, and one can check that $a + c <_0 b + c$, so the claim holds.

It is also easy to show that the claim holds when $j = k$ and $a = 0,\ b \neq 0$. Thus A^* is an ordered abelian group with the ordering defined above.

Let $G = A^*$ and take $(1,0) \in G$ as its distinguished element $1 > 0$. Then

$$\{x \in G : 0 \leq x < 1\} = \{(0,a) : a \in A\},$$

and we have an isomorphism $A \to G_{\text{mod1}}$ of oriented abelian groups given by $a \mapsto (0,a)$. This makes results from the beginning of this section available. When convenient we identify A with the oriented abelian group G_{mod1} via the isomorphism above, and $\mathbb{Z}$ with the ordered subgroup $\mathbb{Z} \cdot 1$ of G via $k \mapsto (k,0) = k \cdot 1$.

Lemma 8.1.2. *For each* $n > 0$ *we have* $|A^*/nA^*| = |\mathbb{Z}/\mathbb{Z} \cap nA^*| \cdot |A/nA|$.

Proof. The surjective group morphism $A^* \to A$ taking (k,a) to a has kernel $\mathbb{Z}$. For $n > 0$, the induced map $A^*/nA^* \to A/nA$ has kernel isomorphic to $\mathbb{Z}/\mathbb{Z} \cap nA^*$. The lemma follows. $\square$

We express the index $|\mathbb{Z}/\mathbb{Z} \cap nA^*|$ in the lemma above in terms of A alone without mentioning A^*, when n is a prime power. Let e range over $\mathbb{N}$. Then

$$\mathbb{Z} \cap p^e A^* = \mathbb{Z} \text{ iff } 1 \in p^e A^* \text{ iff } |A[p^e]| = p^e. \qquad (*)$$

Also $A[p^e]$ is a proper subgroup of $A[p^{e+1}]$ if and only if $|A[p^{e+1}]| = p^{e+1}$. So either $|A[p^e]| = p^e$ for all e, and then we set $e(A,p) := \infty$, or there is a largest e such that $|A[p^e]| = p^e$, in which case we define $e(A,p)$ to be this largest e. So $e(A,p) \in \mathbb{N} \cup \{\infty\}$.

Lemma 8.1.3. *Let* $e \in \mathbb{N}$. *Then*

(1) *if* $e \leq e(A,p)$, *then* $\mathbb{Z} \cap p^e A^* = \mathbb{Z}$, $|A[p^e]| = p^e$ *and* $|A^*/p^e A^*| = |A/p^e A|$;

(2) *if* $e \geq e(A,p)$, *then* $\mathbb{Z} \cap p^e A^* = p^{e-e(A,p)}\mathbb{Z}$, $|A[p^e]| = p^{e(A,p)}$ *and* $|A^*/p^e A^*| = p^{e-e(A,p)}|A/p^e A|$.

Proof. Item (1) follows easily from Lemma 8.1.2 and $(*)$. This gives (2) for $e = e(A,p)$. We continue to prove (2) by induction on e. Suppose (2) holds for a certain $e \geq e(A,p)$. Then

$$p^{e+1-e(A,p)}\mathbb{Z} \ = \ p(\mathbb{Z} \cap p^e A^*) \ \subseteq \ \mathbb{Z} \cap p^{e+1} A^* \ \subseteq \ \mathbb{Z} \cap p^e A^*.$$

The last inclusion is proper: otherwise, $p^{e-e(A,p)} = p^{e+1}a^*$ with $a^* \in A^*$, so $1 = p^{e(A,p)+1}a^* \in p^{e(A,p)+1}A^*$, contradicting the maximality property of $e(A,p)$. Hence $\mathbb{Z} \cap p^{e+1}A^* = p(\mathbb{Z} \cap p^e A^*) = p^{e+1-e(A,p)}\mathbb{Z}$. ☐

Lemma 8.1.4. *Suppose that* $0 < e(A,p) < \infty$. *Then* $[p]A > 1$.

Proof. Let $e = e(A,p)$. Then $A[p^{e-1}]$ is a proper subgroup of $A[p^e] = A[p^{e+1}]$, therefore we can take $a \in A[p^e] \setminus A[p^{e-1}]$. Then $p^{e-1}a \neq 0$ and $p^e a = 0$. It follows that $a \notin pA$: otherwise, $a = pb$ with $b \in A$, so $b \in A[p^{e+1}] = A[p^e]$, so $p^e b = p^{e-1}a = 0$, a contradiction.

☐

8.1.1 Regularly dense oriented abelian groups

We want to define a notion of 'regularly dense' for oriented abelian groups, in analogy with the corresponding notion for ordered abelian groups. First we prove a general lemma on ordered abelian groups.

Lemma 8.1.5. *Let* G *be an ordered abelian group with a distinguished element* $1 > 0$ *such that* $\mathbb{Z}\cdot 1$ *is cofinal in* G, *and the interval* $(0,1) \subseteq G$ *is a nonempty dense linearly ordered set without endpoints. Suppose* S *is a subgroup of* G *such that* $S \cap (0,1)$ *is dense in* $(0,1)$. *Then* S *is dense in* G.

Proof. Note: we do not assume that $1 \in S$. By induction on m, we show that for every $g, h \in G$ with $0 < g < h < 1$, the interval $(m+g, m+h)$ contains an element of S. The case $m = 0$ is just the assumption that $S \cap (0,1)$ is dense in $(0,1)$. Assume the statement holds for a certain m, and take $0 < g < h < 1$. We need to find $x \in S$ such that $m+1+g < x < m+1+h$. Using the induction hypothesis, take

$s \in S \cap (m+h, m+1)$. It suffices to find $y \in S$ such that $m+1+g < y+s < m+1+h$, that is, $m+1+g-s < y < m+1+h-s$. There is such a y since $0 < m+1+g-s < m+1+h-s < 1$. $\square$

We can now prove the following lemma, which says that a certain condition on A is equivalent to A^* being regularly dense as an ordered abelian group.

Lemma 8.1.6. *The following conditions are equivalent:*

(1) A^* *is regularly dense as an ordered abelian group,*

(2) $|A| > 2$ *and for each* p *and all* $a, b \in A$ *with* $a <_0 b$ *there is* $c \in A \setminus \{0\}$ *such that* $ic <_0 (i+1)c$ *for* $i = 1, \ldots, p-1$ *and* $\mathcal{O}(a, pc, b)$.

Proof. Assume (1) and let $a, b \in A$ satisfy $a <_0 b$. Put $x = (0, a)$, $y = (0, b)$, so $0 < x < y < 1$ in A^* where $0 := (0,0)$ and $1 := (1,0)$. Take z in A^* with $x < pz < y$. Then $0 < z < 2z < \cdots < pz < 1$, so $z = (0, c)$ with $c \in A \setminus \{0\}$. One checks easily that then $c <_0 2c <_0 \cdots <_0 pc$ and $\mathcal{O}(a, pc, b)$.

Now assume (2). It is clear that $A \setminus \{0\}$ is a nonempty dense linearly ordered set without endpoints. Then $(0,1) \subseteq A^*$ is a nonempty dense linear ordering without endpoints, and for each p the set $(0,1) \cap pA^*$ is dense in $(0,1)$. Applying the previous lemma with $G = A^*$ and $S = pA^*$, we get that A^* is regularly dense. $\square$

Definition 8.1.7. We say that the oriented abelian group A is *regularly dense*, if condition (2) of Lemma 8.1.6 is satisfied.

Hence A is regularly dense if and only if A^* is regularly dense as an ordered abelian group.

Remark. Suppose that for every p there is nonzero $a \in A$ such that $pa = 0$. Then A is regularly dense iff for every p and all $a, b \in A$ with $a <_0 b$, there is $c \in A$ such that $\mathcal{O}(a, pc, b)$.

We say A is *dense* if it is nontrivial and $<_0$ is a dense linear order without endpoints on $A \setminus \{0\}$. It is easy to see that A is dense iff A^* is dense as an ordered abelian group.

Our main interest is in the infinite subgroups of $\mathbb{S}$, which are the dense subgroups of $\mathbb{S}$. Since $\mathbb{S}^*$ and $\mathbb{R}$ are isomorphic as ordered abelian groups, the dense subgroups of $\mathbb{S}$ are exactly the regularly dense subgroups. Therefore from now on we restrict our attention to regularly dense oriented abelian groups.

Let $\mathcal{L}$ be the language of abelian groups augmented by a ternary relation symbol, recall that $\mathcal{L}_{\text{oab}}$ is the language of ordered abelian groups, and let $\mathcal{L}_{\text{oab}}(1)$ be the language extending $\mathcal{L}_{\text{oab}}$ by a constant symbol 1. When A is an oriented abelian group, we construe it as an $\mathcal{L}$-structure by interpreting the ternary relation symbol as $\mathcal{O}$, and construe A^* as an $\mathcal{L}_{\mathrm{o}}(1)$-structure by interpreting 1 as $(1,0) \in A^*$. Note that then the identification $a \mapsto (0,a) : A \to A^*$ defines the structure A in the structure A^*.

Below elementary equivalences of oriented abelian groups are in the language $\mathcal{L}$, and those of ordered abelian groups are in the language $\mathcal{L}_{\mathrm{o}}(1)$, unless stated otherwise.

Corollary 8.1.8. *Let A, B be regularly dense oriented abelian groups. Then $A \equiv B$ if and only if $A^* \equiv B^*$.*

Proof. The implication $A^* \equiv B^* \Rightarrow A \equiv B$ follows by defining A and B in A^* and B^* as above.

Now let $A \equiv B$. It is shown in [22] that $A^* \equiv B^*$ as $\mathcal{L}_{\mathrm{o}}$-structures if and only if $|A^*/pA^*| = |B^*/pB^*|$ for every p. Thus by Corollary 8.1.3, we get $A^* \equiv B^*$ as $\mathcal{L}_{\mathrm{o}}$-structures. To show $A^* \equiv B^*$, it remains to prove that $(1,0)$ has the same $\mathcal{L}_{\mathrm{o}}$-type (over $\emptyset$) in A^* and B^*. By the quantifier elimination result, Lemma 7.7 from [9], for regularly dense ordered abelian group it is enough to show that

$$1 \in nA \Longleftrightarrow 1 \in nB,$$

for every $n > 0$. This follows from the fact that $|A[n]| = |B[n]|$ for every $n > 0$. ☐

From [22] we have a complete list of invariants for the elementary theory of a regularly dense ordered abelian group, and together with Corollary 8.1.3 and the proof of Corollary 8.1.8 this gives a complete double list of invariants for the elementary theory of a regularly dense oriented abelian group:

Corollary 8.1.9. *Let A and B be regularly dense oriented abelian groups. Then $A \equiv B$ if and only if for every p we have $[p]A = [p]B$ and $e(A,p) = e(B,p)$.*

Next we show that for regularly dense oriented abelian groups A the only relation between the invariants $[p]A$ and $e(A,p)$ is given by Lemma 8.1.4. For every p, let $d_p, e_p \in \mathbb{N}_\infty$ be such that $d_p > 0$ whenever $0 < e_p < \infty$. We proceed to construct an oriented abelian group A such that

$$[p]A = p^{d_p} \text{ and } e(A,p) = e_p \quad \text{for all } p.$$

We do this by building a regularly dense ordered subgroup $G \supseteq \mathbb{Z}$ of the additive group of $\mathbb{R}$ such that $A = G_{\mathrm{mod}1}$ satisfies these conditions. Recall that $\mathbb{Z}_{(p)} = \{\frac{x}{y} \in \mathbb{Q} : x, y \in \mathbb{Z}, p \not| y\}$, the additive group of the localization of $\mathbb{Z}$ at the prime ideal $p\mathbb{Z}$, viewed as a subgroup of the additive group of $\mathbb{Q}$. Put $\frac{1}{p^\infty}\mathbb{Z} := \bigcup_n \frac{1}{p^n}\mathbb{Z}$, and let $H := \sum_p \frac{1}{p^{e_p}}\mathbb{Z} \subseteq \mathbb{Q}$. Note that $[p]H = p$ for $e_p < \infty$, and $[p]H = 1$ for $e_p = \infty$. Let H' be a subgroup of the additive group of $\mathbb{R}$ isomorphic to

$$\bigoplus_{e_p=0} \mathbb{Z}_{(p)}^{d_p} \oplus \bigoplus_{0<e_p<\infty} \mathbb{Z}_{(p)}^{d_p-1} \oplus \bigoplus_{e_p=\infty} \mathbb{Z}_{(p)}^{d_p},$$

such that $H' \cap \mathbb{Q} = \{0\}$. Define

$$G := H \oplus H' \subseteq \mathbb{R}.$$

Then $[p]G = p^{d_p}$ for $0 < e_p$, and $[p]G = p^{d_p+1}$ for $e_p = 0$. It is easy to see that $e(A,p) = e_p$ for every p, where $A := G_{\text{mod}1}$. Hence by Lemma 8.1.3, $[p]A = p^{d_p}$ for every p. Note that A is infinite and it can be identified with a subgroup of $\mathbb{S}$. Thus A is a regularly dense oriented abelian group satisfying the required conditions.

8.1.2 Back-and-forth for regularly dense oriented abelian groups

Let A be an oriented abelian group with a subgroup A' such that $A_{\text{tor}} = A'_{\text{tor}}$, and let $a \in A$. Then we define

$$A'\langle a\rangle_A := \{b \in A : nb = a' + ka \text{ for some } n > 0,\ a' \in A' \text{ and } k \in \mathbb{Z}\}.$$

Note that $A'\langle a\rangle_A$ is the smallest pure subgroup of A containing both A' and a.

Next we obtain an extension procedure for regularly dense oriented abelian groups, using the analogous procedure in [22] for regularly dense ordered abelian groups.

Let A and B be regularly dense oriented abelian groups with $[p]A = [p]B$ for every p. Let $f : A' \to B'$ be an oriented abelian group isomorphism between pure subgroups A' and B' of A and B respectively such that $A'_{\text{tor}} = A_{\text{tor}}$ and $B'_{\text{tor}} = B_{\text{tor}}$. Hence $e(A,p) = e(B,p)$ for every p, and we have an ordered abelian group isomorphism

$$f^* : (A')^* \to (B')^*, \quad (l, a') \mapsto (l, f(a')).$$

Suppose that B is κ-saturated, where $\kappa > |B'|$ is an uncountable cardinal, and let $a \in A \setminus A'$.

Claim. There is $b \in B$ such that there is an oriented group isomorphism $A'\langle a\rangle_A \to B'\langle b\rangle_B$ extending f and taking a to b.

Proof. Note that just because B is κ-saturated, it is not necessarily true that B^* is κ-saturated. So take a κ-saturated elementary extension C^* of B^*, and consider $a^* := (0, a) \in A^*$. Then by the extension procedure mentioned in [22], there is $c^* \in C^*$ such that for all $a' \in (A')^*$, $k \in \mathbb{Z}$, $n > 0$:

(1) $a' < na^* \iff f^*(a') < nc^*$;

(2) $a' + ka^* \in nA^* \iff f^*(a') + kc^* \in nC^*$,

In particular c^* is of the form $(0, c)$, where c is contained in the oriented group extension $C := (C^*)_{\text{mod}1}$ of B. By κ-saturation of B, there is $b \in B$ such that $\text{tp}_C(b|B') = \text{tp}_C(c|B')$. Thus for all $a' \in (A')^*$, $k \in \mathbb{Z}$, $n > 0$:

(1) $a' < na^* \iff f^*(a') < nb^*$;

(2) $a' + ka^* \in nA^* \iff f^*(a') + kb^* \in nB^*$,

where $b^* := (0, b)$. For this b^* we can extend f^* to an ordered group isomorphism $(A')^*\langle a^*\rangle_{A^*} \to (B')^*\langle b^*\rangle_{B^*}$ sending a^* to b^*. This induces an oriented abelian group isomorphism $A'' \to B''$ extending f, where $A'' := ((A')^*\langle a^*\rangle_{A^*})_{\text{mod}1} \subseteq A$ and $B'' := ((B')^*\langle b^*\rangle_{B^*})_{\text{mod}1} \subseteq B$. It is easy to see that A'' is $A'\langle a\rangle_A$, and similarly B'' is $B'\langle b\rangle_B$. □

8.2 The field of real numbers with a subgroup of the circle group

Fix an infinite subgroup Γ of the oriented abelian group $\mathbb{S} \subseteq \mathbb{C}^\times$ with the Mann property. We identify $\mathbb{C}$ with $\mathbb{R}^2$ in the usual way, via

$$z = \alpha + i\beta \mapsto (\alpha, \beta), \qquad (\alpha, \beta \in \mathbb{R}).$$

This makes Γ a subset of $\mathbb{R}^2$. For an element $\alpha = (\alpha_1, \alpha_2)$ of $\mathbb{C}$, we let $\text{Re}(\alpha) := \alpha_1$ and $\text{Im}(\alpha) := \alpha_2$. We put

$$\text{Re}(\Gamma) := \{\text{Re}(\alpha) : \alpha \in \Gamma\} \text{ and } \text{Im}(\Gamma) := \{\text{Im}(\alpha) : \alpha \in \Gamma\}.$$

The *orientation axioms of* Γ are the following: given $\gamma_1, \dots, \gamma_n \in \Gamma$ and a polynomial $Q(X_1, \dots, X_n)$ from $\mathbb{Z}[X_1, \dots, X_n]$, the orientation axiom for $\gamma_1, \dots, \gamma_n, Q$ is

$$Q\big(\text{Re}(\gamma_1), \dots, \text{Re}(\gamma_n)\big) > 0,$$

if this inequality holds in $\mathbb{R}$, and it is

$$Q\big(\text{Re}(\gamma_1), \dots, \text{Re}(\gamma_n)\big) \leq 0$$

otherwise.

Let $\mathcal{L}_{\mathrm{o}}(P,\Gamma)$ be the language of ordered rings augmented by a binary relation symbol P and by a name for each $\gamma \in \Gamma$.

For every linear equation

$$a_1x_1 + \cdots + a_nx_n = 1 \quad (n \geq 2,\ a_1,\ldots,a_n \in \mathbb{Q}^{\times})$$

take a finite list of its nondegenerate solutions in Γ,

$$\gamma_1 = (\gamma_{11},\ldots,\gamma_{1n}),\ldots,\gamma_k = (\gamma_{k1},\ldots,\gamma_{kn}),$$

and let the corresponding *Mann axiom of* Γ be the sentence

$$\forall y \forall z \Big[\big(P(y,z) \wedge \sum_{i=1}^{n} a_i y_i = 1 \wedge \sum_{i=1}^{n} a_i z_i = 0 \wedge \bigwedge_{I} (\sum_{i\in I} a_i y_i \neq 0 \vee \sum_{i \in I} a_i z_i \neq 0)\big) \longrightarrow \bigvee_{j=1}^{k} (y,z) = \gamma_j \Big]$$

in the language $\mathcal{L}_{\mathrm{o}}(P,\Gamma)$, where $y_1,\ldots,y_n,z_1,\ldots,z_n$ are distinct variables, $y = (y_1,\ldots,y_n)$, $z = (z_1,\ldots,z_n)$, $P(y,z)$ abbreviates $P(y_1,z_1) \wedge \cdots \wedge P(y_n,z_n)$, the conjunction $\bigwedge_I$ is over all nonempty proper subsets I of $\{1,\ldots,n\}$, "$\sum_{i=1}^{n} a_i y_i = 1$","$\sum_{i=1}^{n} a_i z_i = 0$" "$\sum_{i \in I} a_i y_i \neq 0$", and "$\sum_{i\in I} a_i z_i \neq 0$" represent certain obvious formulas in the language of rings, and $(y,z) = \gamma_j$ abbreviates

$$y_1 = \mathrm{Re}(\gamma_{j1}) \wedge \cdots \wedge y_n = \mathrm{Re}(\gamma_{jn}) \wedge z_1 = \mathrm{Im}(\gamma_{j1}) \wedge \cdots \wedge z_n = \mathrm{Im}(\gamma_{jn}).$$

Let RCF(Γ) be the $\mathcal{L}_{\mathrm{o}}(P,\Gamma)$-theory whose models are of the form $\big(K, G, (\gamma')_{\gamma\in\Gamma}\big)$ such that

(1) K is a real closed ordered field,

(2) G is a dense subgroup of $\mathbb{S}(K) \subseteq K^2$,

(3) $\gamma \mapsto \gamma' : \Gamma \to G$ is a group morphism,

(4) $\big(K, (\gamma')_{\gamma\in\Gamma}\big)$ satisfies the orientation axioms for Γ,

(5) $\big(K, G, (\gamma')_{\gamma\in\Gamma}\big)$ satisfies the Mann axioms for Γ,

(6) $G_{\mathrm{tor}} = \Gamma_{\mathrm{tor}}$.

Whenever $\big(K, G, (\gamma')_{\gamma\in\Gamma}\big)$ is a model of RCF(Γ), there is an isomorphic copy of the abelian group Γ in G. We identify this copy of Γ with itself, and hence a model of RCF(Γ) will be denoted as $\big(K, G, (\gamma)_{\gamma\in\Gamma}\big)$ or simply $\big(K, G, (\gamma)\big)$. Next we prove Theorem 1.0.5 in a stronger form by characterizing the models of $\mathrm{RCF}(\Gamma)$ up to elementary equivalence.

Theorem 8.2.1. *Let $(K, G, (\gamma))$ and $(L, H, (\gamma))$ be two models of $\mathrm{RCF}(\Gamma)$. Then $(K, G, (\gamma)) \equiv (L, H, (\gamma))$ if and only if $[p]G = [p]H$ for all p, and for all $\gamma \in \Gamma$, and $n > 0$,*

$$\gamma \textit{ is an } n^{\text{th}} \textit{ power in } G \Leftrightarrow \gamma \textit{ is an } n^{\text{th}} \textit{ power in } H.$$

Proof. We only prove the backward direction. So let $(K, G, (\gamma))$ and $(L, H, (\gamma))$ be such that $[p]G = [p]H$ for all p, and for all $\gamma \in \Gamma$, and $n > 0$,

$$\gamma \text{ is an } n^{\text{th}} \text{ power in } G \Leftrightarrow \gamma \text{ is an } n^{\text{th}} \text{ power in } H.$$

We need to prove that $(K, G, (\gamma)) \equiv (L, H, (\gamma))$. We do this by constructing a nonempty back-and-forth system between $(K, G, (\gamma))$ and $(L, H, (\gamma))$. We assume they are κ-saturated with κ an uncountable cardinal.

Let $\mathrm{Sub}(K, G)$ be the collection of $\mathcal{L}_{\mathrm{o}}(P)$-structures (K', G') such that K' is a real closed ordered subfield of K of cardinality less than κ, G' is a pure subgroup of G containing Γ, and $K'(i)$ and $\mathbb{Q}(G)$ are free over $\mathbb{Q}(G')$(as subfields of $K(i)$). Using Lemma 5.13 from [9], we get that if $(K', G') \in \mathrm{Sub}(K, G)$, then $G' = K'(i) \cap G$ and $\mathbb{Q}(G) | \mathbb{Q}(G')$ is regular. Hence $K'(i)$ and $\mathbb{Q}(G)$ are linearly disjoint over $\mathbb{Q}(G')$. Define $\mathrm{Sub}(L, H)$ likewise, and let $\mathcal{I}$ be the collection of isomorphisms between members of $\mathrm{Sub}(K, G)$ and $\mathrm{Sub}(L, H)$ fixing Γ pointwise. We show that $\mathcal{I}$ is a non-empty back-and-forth system. For non-emptiness, let

$$G' := \{g \in G : g^n \in \Gamma \text{ for some } n > 0\}, \quad K' := \mathbb{Q}(\mathrm{Re}(\Gamma))^{\mathrm{rc}} \subseteq K,$$

$$H' := \{h \in H : h^n \in \Gamma \text{ for some } n > 0\}, \quad L' := \mathbb{Q}(\mathrm{Re}(\Gamma))^{\mathrm{rc}} \subseteq L.$$

It is clear that $(K', G') \in \mathrm{Sub}(K, G)$ and $(L', H') \in \mathrm{Sub}(L, H)$ and by the orientation axioms there is an ordered field isomorphism $K' \to L'$ extending the identity map on Γ, which in turn is an isomorphism between (K', G') and (L', H') belonging to $\mathcal{I}$.

Now let $\iota : (K', G') \to (L', H')$ be in $\mathcal{I}$, and $\alpha \in K \setminus K'$.

Case 1: $\alpha \in \mathrm{Re}(G)$. Let $\alpha' \in K$ such that $g := \alpha + i\alpha' \in G$. Then by using the last remark of the previous section, take $h \in H$ and extend $\iota | G'$ to an oriented abelian group isomorphism $G'\langle g \rangle_G \to H'\langle h \rangle_H$. In particular, $\mathrm{Re}(h)$ realizes the cut over $\mathrm{Re}(H')$ corresponding to the cut of α over $\mathrm{Re}(G')$. Moreover we can arrange h in way that $\mathrm{Re}(h)$ realizes the cut over L' corresponding to the cut of α over K'. Hence

$K'(\alpha)^{\mathrm{rc}}$ and $L'(h)^{\mathrm{rc}}$ are isomorphic as ordered fields through a map extending ι, sending α to $\mathrm{Re}(h)$. This is an isomorphism of $\mathcal{L}_{\mathrm{o}}(P)$-structures $(K'(\alpha)^{\mathrm{rc}}, G'')$ and $(L'(\mathrm{Re}(h))^{\mathrm{rc}}, H'')$. We need to check that $K'(\alpha, i)^{\mathrm{rc}}$ and $\mathbb{Q}(G)$ are free over $\mathbb{Q}(G'')$. It follows by usual arguments. (Note that this case also covers the situation that $\alpha \in \mathrm{Im}(G)$.)

Case 2: $\alpha \in K'(\mathrm{Re}(G) \cup \mathrm{Im}(G))^{\mathrm{rc}}$. We apply the previous case several times.

Case 3: $\alpha \notin K'(\mathrm{Re}(G) \cup \mathrm{Im}(G))^{\mathrm{rc}}$. Then take $\beta \in L$ such that $\beta \notin L'(\mathrm{Re}(H) \cup \mathrm{Im}(H))^{\mathrm{rc}}$ realizing the type over L' corresponding to the type of α over K'. So $K'(\alpha)^{\mathrm{rc}}$ and $L'(\beta)^{rc}$ are isomorphic as ordered fields. In fact, it is an $\mathcal{L}_{\mathrm{o}}(P)$-isomorphism extending ι. This finishes the proof. □

Example. Here we present an example similar to the examples at the end of Chapter 3. It shows that the arguments of the current chapter do not go through when the real field $\mathbb{R}$ is replaced by its polynomially bounded expansion $\mathbb{R}_{\mathrm{an}}$. Let

$$f(x) := \begin{cases} \frac{\arccos(x)}{\pi} & \text{if } -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mathrm{Re}(\mathbb{U}) := \big\{ x \in [-1,1] : \text{ there is } y \in \mathbb{R} \text{ such that } (x,y) \in \mathbb{U} \big\} \subseteq \mathbb{R}.$$

Then $\mathrm{Re}(\mathbb{U})$ is definable in $(\mathbb{R}, \mathbb{U})$, and f is definable in $\mathbb{R}_{\mathrm{an}}$. The image of $\mathrm{Re}(\mathbb{U})$ under f is $\mathbb{Q} \cap [0,1]$. So by taking reciprocals, we can define the set $\mathbb{Q} \subseteq \mathbb{R}$ in $(\mathbb{R}, \mathbb{U}, f)$. Therefore the set $\mathbb{Z}$ is definable in $(\mathbb{R}, \mathbb{U}, f)$, hence.in $(\mathbb{R}_{\mathrm{an}}, \mathbb{U})$.

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