A TALK AROUND COMPLEX EXPONENTIATION

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1. Introduction to Pseudoexponentiation and Marker's work

We consider the expansion $(\mathbb{C}, +, \cdot, 0, 1, \exp)$ of the field of complex numbers by the exponentiation; we generally denote it as $(\mathbb{C}, \exp)$. This structure is not as well-behaving as the real field with exponentiation is. One reason is that the ring of integers is definable in it, but it is not a big problem as one can show $(\mathbb{C}, \mathbb{Z})$ is still not too bad model theoretically (see [1] and [4]). However, one can show that it is not model complete (see [2]). The big question is whether every subset of $\mathbb{C}$ definable in $(\mathbb{C}, \exp)$ is either countable or co-countable. This would, in particular, show that $\mathbb{R}$ is not definable in it and hence we do not get the whole of projective hierarchy directly.

To handle $(\mathbb{C}, \exp)$, Zilber introduced an $\mathcal{L}_{\omega_1,\omega}(Q)$-sentence Φ that has exactly one model in every uncountable cardinality, where Q is the quantifier 'there exists uncountably many' (see [5]). Of course the question is whether $(\mathbb{C}, \exp)$ is the unique model of cardinality $2^{\aleph_0}$. We present Φ in the form of several axiom schemes. In what follows K always represents a field. First of all, for the ease of notation put $G_n(K) = K^n \times (K^\times)^n$.

Now we are ready to state the axiom schemes. Let $\mathcal{L} = \{+, \cdot, 0, 1, E\}$.

(1) K is an algebraically closed field of characteristic 0 and E is a group homomorphism between $(K, +)$ and $(K^\times, \cdot)$ and there is a transcendental $\alpha \in K$ such that the kernel of E is $\mathbb{Z}\alpha$.

(2) (Schanuel's conjecture) For every $\alpha_1, \dots, \alpha_n \in K$ linearly independent over $\mathbb{Q}$, the transcendence degree of $\big(\alpha_1, \dots, \alpha_n, E(\alpha_1), \dots, E(\alpha_n)\big)$ is at least n.

(3) (Strong Exponential Closure) For every irreducible, free, normal algebraic set $V \subseteq G_n(K)$ and finite set A, there is a generic point of V over A of the form $\big(\vec{\alpha}, E(\vec{\alpha})\big)$.

(4) (Countable Closures) For every irreducible, free, normal algebraic set $V \subseteq G_n(K)$ over A with $\dim V = n$, the set $\big\{\big(\vec{\alpha}, E(\vec{\alpha})\big) \in V : \text{generic over } A\big\}$ is countable.

Schanuel's conjecture is still a conjecture for $(\mathbb{C}, \exp)$ and Zilber has shown that 'Countable Closures' holds in $(\mathbb{C}, \exp)$ (see [5]). Here we are proceeding to show that an instance of 'Strong Exponential Closure' holds in $(\mathbb{C}, \exp)$ if we assume the Schanuel's conjecture.

Under the assumption of Schanuel's conjecture, Marker has proven the following, which is the simplest case of 'Strong Exponential Closure' (see [2]).

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Theorem 1.1. *Let $P(X,Y) \in \mathbb{Q}^{\mathrm{ac}}[X,Y]$ be irreducible and depend on both X and Y. Then there are infinitely many roots of $P\big(z, \exp(z)\big) = 0$ in $\mathbb{C}$ that are algebraically independent over $\mathbb{Q}$.*

Later he extended this to $P(X,Y) \in \mathbb{Q}(a)[X,Y]$, where

$$\operatorname{trdeg}_{\mathbb{Q}}(a_1,\dots,a_n) = \operatorname{trdeg}_{\mathbb{Q}}(a_1,\dots,a_n,\exp(a_1),\dots,\exp(a_n)) = 1.$$

The main lemma for doing this is the following.

Lemma 1.2. *For irreducible $P(X,Y) \in \mathbb{Q}(a_1,\dots,a_n)[X,Y]$ that depends on both X and Y, there are only finitely many zeros of $P(z,\exp(z))$ in $\mathbb{Q}(a_1,\dots,a_n)$.*

Marker comments that we can get rid of the assumption on transcendence degree if we could prove this lemma for arbitrary extension of $\mathbb{Q}$ of finite degree. This is what I'll do today.

2. Reducing to a finite rank group

Throughout $K = \mathbb{Q}(\alpha_1,\dots,\alpha_s)^{\mathrm{ac}}$ and put $\Gamma = \exp(K)$. Then the transcendence degree of K over $\mathbb{Q}$ is less than s, say d. (*We don't really need that K is algebraically closed; for the proofs, it is enough that K is contained in $\mathbb{Q}(\alpha_1,\dots,\alpha_s)^{\mathrm{ac}}$ or that K is a finitely generated field. But with the current assumption the final result is stronger.*)

Note that Γ has a torsion element if and only if $\pi \in K$; and hence once Γ has one torsion element it has all the others, i.e. $\mathbb{U} \leq \Gamma$. A subgroup Δ of Γ is called *radical* if it is a pure subgroup and it has all the torsions of Γ. For a subset A of Γ, the *radical closure* in Γ of the group generated by A is denoted by $\langle A\rangle$; that is

$$\langle A\rangle := \{\gamma \in \Gamma : \gamma^n \text{ is in the subgroup of } \Gamma \text{ generated by } A \text{ for some } n > 0\}.$$

So $\langle A\rangle$ is the smallest radical subgroup of Γ containing A.

From now on we always assume Schnauel's Conjecture. The next lemma is not essential for the later results, but it looks useful, so I include it here.

Lemma 2.1. *The rank of the group $\Gamma \cap K^\times$ is at most d.*

Proof. Let let $\beta_1,\dots,\beta_d,\beta_{d+1} \in K$ with $\exp(\beta_1),\dots,\exp(\beta_d),\exp(\beta_{d+1}) \in K^\times$. Then the transcendence degree

$$\operatorname{trdeg}_{\mathbb{Q}}(\beta_1,\dots,\beta_{d+1},\exp(\beta_1),\dots,\exp(\beta_{d+1}))$$

is at most d. Thus using Schanuel's conjecture we get that $\exp(\beta_1),\dots,\exp(\beta_{d+1})$ are multiplicatively dependent. $\square$

On the basis of this lemma, take $\beta_1,\dots,\beta_t \in K$ with $t \leq d$ such that $\langle \Gamma \cap K^\times\rangle = \langle \exp(\beta_1),\dots,\exp(\beta_t)\rangle$. Put $\vec{\alpha} = (\alpha_1,\dots,\alpha_s)$ and $\vec{\beta} = (\beta_1,\dots,\beta_t)$ and let e be the transcendence degree of $K(\exp(\vec{\alpha}),\vec{\beta},\exp(\vec{\beta}))$ over $\mathbb{Q}$. Note that

$$d \leq e \leq d + s + t. \tag{2.1}$$

We first consider the solutions in Γ of linear equations with coefficients from K. Vaguely speaking, we are aiming to show that such solutions are determined by finite information. For this we use the following result from [1].

Lemma 2.2. *Let G be a subgroup of $\mathbb{C}^\times$ with a radical subgroup H and let L a subfield of $\mathbb{C}$. Then the following two conditions are equivalent:*

(1) *for any* $c_1, ..., c_m \in L^\times$ *the equation* $c_1x_1 + \cdots + c_mx_m = 1$ *has the same nondegenerate solutions in* H *as in* G*;*
(2) *whenever* $g_1, ..., g_n \in G$ *are multiplicatively independent over* H*, they are algebraically independent over* $L(H)$*.*

Now we are ready to state and prove the main result of this section.

Proposition 2.3. *There is a radical subgroup* Γ^* *of* Γ *of finite rank containing* $\Gamma \cap K^\times$ *such that for every* $a_1, \ldots, a_k \in K$ *the equation* $a_1x_1 + \cdots + a_kx_k = 1$ *has the same non-degenerate solutions in* Γ^* *as in* Γ*.*

Proof. We use Lemma 2.2 (by taking H, G and L to be Γ^*, Γ and K respectively). is pure in Γ and each root of unity in Γ is already in $\Gamma \cap K^\times$. We need is to construct a group Γ^* and to check the following.

$(*)$ Let $\gamma_1, \ldots, \gamma_m \in \Gamma$ be algebraically dependent over $K(\Gamma^*)$. Then they are multiplicatively dependent over Γ^*.

If $(*)$ holds with $\langle \Gamma \cap K^\times \rangle$ in the place of Γ^*, then we are done. So assume that there are $b_1, \ldots, b_{m(1)} \in K$ such that $\exp(b_1), \ldots, \exp(b_{m(1)})$ are algebraically dependent over $K(\langle \Gamma \cap K^\times \rangle)$ and multiplicatively independent over $\langle \Gamma \cap K^\times \rangle$. Let $\Gamma^{(1)}$ be $\langle (\Gamma \cap K^\times) \cup \{\exp(b_1), \ldots, \exp(b_{m(1)})\} \rangle$.

Let $b_{m(1)+1}, \ldots, b_{m(2)} \in K$ such that $\exp(b_{m(1)+1}), \ldots, \exp(b_{m(2)})$ are algebraically dependent over $K(\Gamma^{(1)})$. Consider the following degree

$$\mathrm{trdeg}_{\mathbb{Q}}(\vec{\alpha}, \vec{\beta}, b_1, \ldots, b_{m(1)}, b_{m(1)+1}, \ldots, b_{m(2)}, \exp(\vec{\alpha}), \exp(\vec{\beta}), \exp(\vec{b})).$$

This degree is at most $e + m(2) - 2$. Compare it with $s + t + m(2)$. If this difference, $s + t - e + 2$, is strictly bigger than 0, then we are done. If not add $\exp(b_{m(1)+1}), \ldots, \exp(b_{m(2)})$ on top of $\Gamma^{(1)}$ to get $\Gamma^{(2)}$. By the (2.1), we can continue doing this at most d many times. Thus $\Gamma^* := \Gamma^{(d+1)}$ is the group we are looking for. □

3. A fact on polynomial-exponential equations

Let $P(X, Y) \in K[X, Y]$ be irreducible and depend both on X and Y. We are proceeding to show that the set

$$V(P) := \{z \in K : P(z, \exp(z)) = 0\}$$

is finite. We use the following special case of the main result of [3]:

Theorem 3.1. *Let* $Q_i(X_1, \ldots, X_k) \in \mathbb{Q}^{\mathrm{ac}}[X_1, \ldots, X_k]$ *and let* $\vec{\alpha}_i \in ((\mathbb{Q}^{\mathrm{ac}})^\times)^k$ *for* $i = 1, \ldots, l$*. If the group* $G := \{\vec{m} \in \mathbb{Z}^k : \vec{\alpha}_i^{\vec{m}} = \vec{\alpha}_j^{\vec{m}}$ *for every* $i, j\}$ *is trivial, then the set*

$$\Big\{\vec{m} \in \mathbb{Z}^k : \sum_{i=1}^{l} Q_i(\vec{m})\vec{\alpha}_i^{\vec{m}} = 0\Big\}$$

is finite.

4. Reducing to integer solutions

Write $P(X,Y) = \sum_{i=0}^{q} P_i(X)Y^i$ and let $I := \{i \in \{0,1,\dots,q\} : P_i(X) \neq 0\}$. Since $P(X,Y)$ is irreducible we have that $0 \in I$. Now the set

$$\{z \in K : P_i(z) = 0 \text{ for some } i \in I\}$$

is finite. Thus it suffices to show that

$$W(P) := V(P) \setminus \{z \in K : P_i(z) = 0 \text{ for some } i \in I\}$$

is finite. So take $z \in K$ such that $P(z, e^z) = 0$ and $P_i(z) \neq 0$ for every $i \in I$, and consider the linear equation

$$\sum_{i \in I} -\frac{P_i(z)}{P_0(z)} Z_i = 1. \tag{4.1}$$

The tuple $(\exp(z)^i)_{i \in I}$ is a solution of this equation. Moreover we may assume that it is a nondegenerate solution (*because somehow knowing one coordinate is enough*). Now by the previous proposition a nondegenerate solution $\gamma = (\gamma_i)_{i \in I}$ of this equation in Γ must be in Γ^*.

Now write $\Gamma^* = \langle \exp(a_1), \dots, \exp(a_p) \rangle$ with $a_1, \dots, a_p \in K$ and $\exp(a_1), \dots, \exp(a_p)$ are multiplicatively independent. Also write $z = m_1 a_1 + \cdots + m_p a_p + n 2\pi\sqrt{-1}$. So

$$\sum_{i \in I} Q_i(m_1, \dots, m_p, n) \exp(a_1)^{i m_1} \cdots \exp(a_p)^{i m_p} = 0,$$

where $Q_i(X)$ is a polynomial in $\mathbb{Q}(a_1, \dots, a_p, \pi\sqrt{-1})$ depending on $P_i(X)$, a_j's and $\exp(a_j)$'s.

Now using Theorem 3.1, we conclude that there are only finitely many such $(m_1, \dots, m_p, n) \in \mathbb{Z}^{p+1}$ and hence finitely many $z \in K$ with $P(z, e^z) = 0$.

References

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