# A TALK ON 'SMALL' MULTIPLICATIVE SUBGROUPS OF FIELDS

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## 1. Statements of the main results

We first state the main theorems without defining the Mann property. However, one can think a multiplicative subgroup of a field with the Mann property as '*small*' enough so that the addition does not leave a trace on the group. For instance, all the subgroups of $\mathbb{C}^\times$ of finite rank have the Mann property.

From now on $K$ is a field, and $G$ is a subgroup of $K^\times$. Today I will mostly concentrate the algebraically closed case.

**Proposition 1.1.** *For algebraically closed $K$, the following are equivalent:*

1. *$G$ has the Mann Property;*
2. *for every algebraic set $V \subseteq K^n$ its trace $V \cap G^n$ is a finite union of cosets of subgroups of $G^n$;*
3. *for every $X \subseteq K^n$ that is definable in $(K, G)$ its trace $X \cap G^n$ is definable in the group $G$.*

Let $\Gamma$ be a subgroup of $\mathbb{C}^\times$ with the Mann property.

**Theorem 1.2.** *Let $K$ be an algebraically closed field of characteristic $0$, let $G$ be a subgroup of $K^\times$, and let a map $\gamma \mapsto \gamma' : \Gamma \to G$ be given. Then $\big(K, G, (\gamma')_{\gamma\in\Gamma}\big) \equiv \big(\mathbb{C}, \Gamma, (\gamma)_{\gamma\in\Gamma}\big)$ if and only if*

- $\big(G, (\gamma')_{\gamma\in\Gamma}\big) \equiv \big(\Gamma, (\gamma)_{\gamma\in\Gamma}\big)$ *as groups with distinguished elements;*
- $\big(K, G, (\gamma')_{\gamma\in\Gamma}\big)$ *satisfies the Mann axioms of $\Gamma$.*

I will explain 'Mann property', and 'Mann axioms for $\Gamma$' in a bit.

## 2. Smallness

Below, $\mathcal{L}$ is a language, $\mathcal{M} = (M, \dots)$ is an $\mathcal{L}$-structure, and $G \subseteq M$. If $f(G^m) = M$ for some $m, n$ and some $f : M^m \xrightarrow{n} M$ definable in $\mathcal{M}$, we call $G$ *large in $\mathcal{M}$*, and otherwise we call $G$ *small in $\mathcal{M}$*. In particular, if $M$ is infinite and $|G| < |M|$, then $G$ is small in $\mathcal{M}$. (It came to our attention that Casanovas and Ziegler have another notion of *small*. It is easy to see that their notion is equivalent to ours for strongly minimal $\mathcal{M}$.)

We can now state a generalization of a theorem of Keisler on pairs of algebraically closed fields:

**Proposition 2.1.** *Let $T$ be an $\mathcal{L}$-theory with QE whose models are infinite and strongly minimal. Let $\mathcal{M}$ and $\mathcal{N}$ be models of $T$ with substructures $\mathcal{G} = (G, \dots)$ and $\mathcal{H} = (H, \dots)$, such that*

(i) *$G$ is small in $\mathcal{M}$ and $H$ is small in $\mathcal{N}$;*
(ii) *$\mathcal{G} \equiv \mathcal{H}$ and $\mathcal{M} \equiv \mathcal{N}$.*

*Then $(\mathcal{M}, G) \equiv (\mathcal{N}, H)$.*

**Remark.** Suppose $F$ is a subfield of the algebraically closed field $K$. Then $F$ is large in $K$ if and only if $F = K$, or $F$ is a real closed field and $[K : F] = 2$. This follows from a theorem of E. Artin.

## 3. Algebraically closed fields with a multiplicative set

From now on $\mathcal{L} = \{0, 1, +, -, \cdot\}$ is the language of rings. We shall also use its sublanguage $\mathcal{L}_{\mathrm{m}} := \{1, \cdot\}$ of multiplicative monoids.

Throughout this section $K$ is an algebraically closed field, with prime field $\mathbb{F}$, and $G$ is a *multiplicative set in* $K$, that is, $G$ is a subset of $K$ that contains 1 and is closed under multiplication. (For example, any subring of $K$ is a multiplicative set in $K$.) We consider $G$ as an $\mathcal{L}_{\mathrm{m}}$-structure in the obvious way. The *addition* of $K$ also leaves a trace on $G$, and to deal with that we extend $\mathcal{L}_{\mathrm{m}}$ to the language

$$\mathcal{L}_{\mathrm{m}}(\Sigma) := \{1, \cdot\} \cup \{\Sigma_k : k \in \mathbb{Z}^n, n = 1, 2, \dots\}$$

of *multiplicative monoids with additive relations*: here $\Sigma_k$ is an $n$-ary relation symbol for $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$. We expand the monoid $G$ to an $\mathcal{L}_{\mathrm{m}}(\Sigma)$-structure $G(\Sigma)$ by interpreting $\Sigma_k$ as the $n$-ary relation

$$\{(g_1, \dots, g_n) \in G^n : k_1 g_1 + \cdots + k_n g_n = 0\}$$

on $G$. As a special case of Proposition 2.1 we have:

**Corollary 3.1.** *If $G$ is small in $K$, then* $\mathrm{Th}(K, G)$ *is completely determined by* $\mathrm{Th}(G(\Sigma))$.

In other words, if $G$ is small in $K$, and $H$ is a small multiplicative set in an algebraically closed field $L$, then

$$(K, G) \equiv (L, H) \iff G(\Sigma) \equiv H(\Sigma).$$

To see how this follows from 2.1, note that algebraically closed fields can be construed as $\mathcal{L}_{\mathrm{m}}(\Sigma)$-structures, that their theory in this language admits QE, and that the substructures of an algebraically closed field $L$ viewed as $\mathcal{L}_{\mathrm{m}}(\Sigma)$-structure are exactly the structures $H(\Sigma)$ where $H$ is a multiplicative set in $L$. Also, the characteristic of an algebraically closed field is determined by the sentences $\Sigma_k(1)$ that it satisfies where $1 \le k \in \mathbb{Z} = \mathbb{Z}^1$.

In the next result $G$ is not assumed to be small in $K$.

**Proposition 3.2.** *Every subset of $G^n$ definable in $(K, G)$ is definable in the structure $G(\Sigma)$.*

**Remarks.**

(1) The proposition deals with a special case of the situation considered in Proposition 3.1 of Pillay in Bouscaren volume, but is a bit stronger in that special case.
(2) In case $G$ is a subring of $K$, Proposition 3.2 says that the subsets of $G^n$ definable in $(K, G)$ are definable in the ring $G$.
(3) The proposition fails badly when $K$ is replaced by the field $\mathbb{R}$. For example, *every* subset of $\mathbb{Z}$ is definable in $(\mathbb{R}, \mathbb{Z})$.

The following are consequences of a back-and-forth argument:

**Corollary 3.3.** *Suppose $G$ is small in $K$ and the $\mathcal{L}_{\mathrm{m}}(\Sigma)$-structure $G(\Sigma)$ is $\omega$-stable. Then the $\mathcal{L}(U)$-structure $(K, G)$ is $\omega$-stable.*

In a similar way we obtain:

**Corollary 3.4.** *If $G$ is small in $K$ and $G(\Sigma)$ is superstable (stable), then $(K, G)$ is superstable (respectively, stable).*

**Corollary 3.5.** *Let $G$ be small in $K$, and let $K'$ be an algebraically closed subfield of $K$ with a multiplicative subset $G'$ that is small in $K'$. Suppose $G' \subseteq G$ with $G'(\Sigma) \preceq G(\Sigma)$, and $K'$ and $\mathbb{F}(G)$ are free over $\mathbb{F}(G')$ in $K$. Then $(K', G') \preceq (K, G)$.*

## 4. The Mann Property

Throughout this section, $K$ is a field, $E$ is a subfield of $K$, and $G$ is a subgroup of the multiplicative group $K^\times$ of $K$. Consider a linear equation

$$(*) \qquad a_0 = a_1x_1 + \cdots + a_nx_n,$$

where $a_0, a_1, \ldots, a_n \in K$. A *solution* of $(*)$ in $G$ is a tuple $(g_1, \ldots, g_n) \in G^n$ such that $a_0 = a_1g_1 + \cdots + a_ng_n$, and such a solution is said to be *nondegenerate* if $\sum_{i\in I} a_ig_i \neq 0$ for each non-empty proper subset $I$ of $\{1, \ldots, n\}$. In the homogeneous case $a_0 = 0$ the set of solutions in $G$ and the set of nondegenerate solutions in $G$ are both unions of orbits with respect to the action of $G$ on $G^n$ defined by $g(g_1, \ldots, g_n) = (gg_1, \ldots, gg_n)$.

Let us say that $G$ has the *Mann property over $E$* if each equation

$$a_1x_1 + \cdots + a_nx_n = 1 \qquad (n \geq 2,\ a_1, \ldots, a_n \in E^\times)$$

has only finitely many nondegenerate solutions in $G$. In the case $E = \mathbb{F}$ we just say that $G$ has the Mann property. This terminology is temporary, since the Mann property will be shown to be equivalent to the Mann property over $E$. Consider a homogeneous equation

$$a_1x_1 + \cdots + a_nx_n = 0, \qquad (a_1, \ldots, a_n \in E^\times, n \geq 1).$$

Let $S \subseteq G^n$ be its set of solutions in $G$ and $S_{\mathrm{nd}}$ its subset of nondegenerate solutions in $G$. Then

$$S_{\mathrm{nd}} \ = \bigcup_{(g_1,\ldots,g_{n-1})\in S'} (g_1, \ldots, g_{n-1}, 1)G \ \subseteq\ S,$$

where $S'$ is the set of nondegenerate solutions of the equation

$$a_1y_1 + \cdots + a_{n-1}y_{n-1} = -a_n$$

in $G$. If $G$ has the Mann property over $E$, then $S'$ is finite, so $S_{\mathrm{nd}} \subseteq G^n$ is then defined in $G$ by the positive quantifier-free formula

$$\bigvee_{(g_1,\dots,g_{n-1})\in S'} x_1 = g_1x_n \wedge \cdots \wedge x_{n-1} = g_{n-1}x_n$$

in the language of $G$-sets. Here we view $G$ as a $G$-set by the (left) action $(g,h) \mapsto gh$. Considering also degenerate solutions of the above homogeneous equation, and using the description above inductively, we obtain:

**Corollary 4.1.** *If $G$ has the Mann property over $E$, then $S \subseteq G^n$ is defined in $G$ by a positive quantifier-free formula in the language of $G$-sets.*

In view of Proposition 3.2 this yields (1) ⇒ (3) of Proposition 1.1:

**Corollary 4.2.** *If $K$ is algebraically closed, $G$ has the Mann property and $X \subseteq K^n$ is definable in $(K,G)$, then $X \cap G^n$ is definable in the group $G$.*

Having the Mann property is not dependent on the field that $G$ lives in:

**Proposition 4.3.** *Suppose that $G$ has the Mann property. Then $G$ has the Mann property over $K$.*

**Mann implies Mordell-Lang.** In this subsection $K$ is algebraically closed. For any $n$-tuple $k = (k_1,\dots,k_n) \in \mathbb{Z}^n$, consider the *character*

$$\chi_k : \left(K^\times\right)^n \to K^\times, \quad \chi_k(x_1,\dots,x_n) := x_1^{k_1}\cdots x_n^{k_n}.$$

This is a multiplicative group homomorphism. For any $d \in \mathbb{N}$, let $\mathcal{D}(n,d)$ be the finite collection of subgroups of $\left(K^\times\right)^n$ that are intersections of kernels of characters $\chi_k$ with $|k| = |k_1| + \cdots + |k_n| \le d$. The following yields (1) ⇒ (2) of Proposition 1.1.

**Proposition 4.4.** *Let $f_1,\dots,f_m \in K[X_1,\dots,X_n]$ have degree $\le d$, and let*

$$V = \{x \in K^n : f_1(x) = \cdots = f_m(x) = 0\}.$$

*Suppose $G$ has the Mann property. Then $V \cap G^n$ is a finite union of cosets of subgroups $D \cap G^n$ of $G^n$ with $D \in \mathcal{D}(n,d)$.*

*Proof.* The intersection of finitely many cosets of such subgroups is either empty or again a coset of such a subgroup. Hence we may (and shall) assume that $m = 1$. Put $f := f_1$, and write $f = \sum_{i\in I} a_iX^i$ where all $a_i \in K$ and $I$ is the set of multi-indices $i = (i_1,\dots,i_n) \in \mathbb{N}^n$ with $|i| = i_1 + \cdots + i_n \le d$. By Corollary 4.1 the set

$$\{y \in G^{|I|} : \sum_{i\in I} a_iy_i = 0\}$$

is a finite union of finite intersections of subsets of $G^{|I|}$ of the form

$$\{y \in G^{|I|} : gy_i = y_j\}$$

with $g \in G$ and $i, j \in I$. It remains to observe that for such $g, i, j$ the set

$$\{x \in G^n : \ g\chi_i(x) = \chi_j(x)\}$$

is a coset of the subgroup $D \cap G^n$ of $G^n$ where $D$ is the kernel of $\chi_{i-j}$. ☐

The proof of the fact that Mordell-Lang implies Mann uses some algebraic group theory, hence I will just state it without a proof.

**Proposition 4.5.** *Suppose that for all $a_1, \ldots, a_n \in \mathbb{F}^\times$ the set of solutions of $a_1x_1 + \cdots + a_nx_n = 1$ in $G$ is a boolean combination of subgroups of $G^n$. Then $G$ has the Mann property.*

The following lemma is quite technical, however it presents the idea in the core of the Mann property.

**Lemma 4.6.** *Let $\Gamma$ be a subgroup of $G$ such that for all $a_1, \ldots, a_n \in E^\times$ the equation $a_1x_1 + \cdots + a_nx_n = 1$ has the same nondegenerate solutions in $\Gamma$ as in $G$. Then we have for any $g, g_1, \ldots, g_n \in G$:*

1. *if $g$ is algebraic over $E(\Gamma)$ of degree $d$, then $g^d \in \Gamma$;*
2. *if $g_1, \ldots, g_n$ are algebraically dependent over $E(\Gamma)$, then they are multiplicatively dependent over $\Gamma$.*

## 5. Algebraically closed fields with a multiplicative group having the Mann Property

*In this section $K$ is an algebraically closed field and $G$ is a subgroup of $K^\times$ with the Mann property.*

**Smallness.** In order to use the results in Section 3 we have to show that $G$ is small in $K$. We shall derive this from the fact that no infinite field is interpretable in any abelian group. This fact follows from three other results:

1. Each abelian group is one-based.
2. If a group is interpretable in a one-based structure, then it has an abelian subgroup of finite index.
3. if $E$ is an infinite field, then the group $\mathrm{SL}_2(E)$ does not have an abelian subgroup of finite index.

**Lemma 5.1.** *$G$ is small in $K$.*

*Proof.* Suppose towards a contradiction that $G$ is large in $K$. Then we have $K = f(G^m)$ where $f : K^m \longrightarrow K$ is definable in $K$. Consider the equivalence relation $E$ on $K^m$ defined by

$$aEb \iff f(a) = f(b).$$

By Proposition 3.2 the equivalence relation $E_G := E \cap G^{2m}$ on $G^m$ is definable in the group $G$. The restriction $f|G^m : G^m \to K$ induces a bijection $G^m/E_G \to K$. Using again Proposition 3.2, one checks easily that the addition and multiplication of $K$ correspond under this bijection to binary operations on $G^m/E_G$ that are definable in the many-sorted structure $G^{\mathrm{eq}}$ where $G$ is considered as a group. This contradicts the fact that no infinite field is interpretable in any abelian group. $\square$

In combination with Lemmas 3.3, 3.4 and Corollary 4.1 this yields:

**Corollary 5.2.** *The $\mathcal{L}(U)$-structure $(K, G)$ is stable. If the group $G$ is $\omega$-stable (superstable), then $(K, G)$ is $\omega$-stable (respectively, superstable).*

Here is another consequence (not used later) of the non-interpretability of infinite fields in abelian groups. Put

$$G_0 := G \cup \{0\} \subseteq K,$$
$$G^{+n} := G_0 + \cdots + G_0 \subseteq K \quad \text{(with } n \text{ summands).}$$

Note that $G_0 = G^{+1} \subseteq G^{+2} \subseteq G^{+3} \subseteq \ldots$.

**Corollary 5.3.** *If $G$ is infinite, then the increasing sequence of sets $\big(G^{+n}\big)$ is strictly increasing.*

As an application, for $G = 2^{\mathbb{Z}}3^{\mathbb{Z}} \cup -2^{\mathbb{Z}}3^{\mathbb{Z}}$, there is no $n$ such that all rational numbers are of the form $a/b$ with $a, b \in G^{+n}$; we do not know how to prove this particular fact other than by the argument above.

**More on the $\omega$-stable case.** Let $H$ be an infinite subgroup of the multiplicative group of a field. Then $H$ has only finitely many elements of any given finite order, so by a theorem of Macintyre, the group $H$ is $\omega$-stable if and only if there is an infinite divisible subgroup $D$ of $H$ and a finite subgroup $B$ of $H$ such that $H = DB$ and $D \cap B = \{1\}$. For such $D$ and $B$ we have $H^{[d]} = D$ for each positive integer $d$ that is a multiple of $|B|$, so the subgroup $D$ does not depend on the particular product decomposition chosen; hence $D$ is a definable subgroup of the group $H$. By Szmielew's quantifier simplification for abelian groups, $D$ has no infinite proper subgroups definable in the group $H$. Thus, for $\omega$-stable $H$, the subgroup $D$ is the *connected component* of the group $H$, and $\mathrm{MR}(H^n) = n$ for each $n$, where the Morley rank is with respect to the theory of the group $H$. We use these observations for $H := G$ to prove the following:

**Theorem 5.4.** *Suppose $G$ is infinite and $\omega$-stable. Then $\mathrm{MR}(K) = \omega$, where the Morley rank is with respect to the $\omega$-stable theory $\mathrm{Th}(K, G)$.*

**Definable sets.** With a mild assumption on $G$ the definable relations in $(K, G)$ are boolean combinations of existentially definable relations. To formulate this precisely, recall that $G^{[d]}$ denotes the subgroup of $d^{\text{th}}$ powers in $G$. Let $\mathcal{L}(K)$ be the language of rings augmented by names for the elements of $K$, and let $x = (x_1, \ldots, x_m)$ be a tuple of distinct variables.

**Proposition 5.5.** *Suppose* $G/G^{[d]}$ *is finite for each integer* $d > 0$. *Then every subset of* $K^m$ *definable in* $(K, G)$ *is a boolean combination of subsets of* $K^m$ *defined by formulas* $\exists y\big(U(y) \wedge \phi(x, y)\big)$ *where* $\phi(x, y)$ *is a quantifier-free* $\mathcal{L}(K)$*-formula.*

**Elementary classification within the Γ-family.** Let $E$ be a field and $\Gamma$ a subgroup of $E^\times$ with the Mann property. All multiplicative groups that contain $\Gamma$ and satisfy the Mann axioms of $\Gamma$ are treated below as members of the same family, the $\Gamma$-family. For example, if $\Gamma = \{1\} \subseteq \mathbb{Q}^\times$, each subgroup of $\mathbb{C}^\times$ generated by algebraically independent elements belongs to the $\Gamma$-family. Likewise, with $E = \mathbb{Q}$ and $\Gamma = 2^{\mathbb{Z}}$, the subgroups $2^{\mathbb{Z}}$ and $2^{\mathbb{Q}}$ of $\mathbb{C}^\times$ belong to the $\Gamma$-family. The formal setting is as follows.

Let $\mathcal{L}(U, \Gamma)$ be the language of rings augmented by a unary relation symbol $U$, and by a name (constant symbol) $\gamma$ for each element $\gamma \in \Gamma$. Let $\mathrm{ACF}(\Gamma)$ be the theory in the language $\mathcal{L}(U, \Gamma)$ whose models are the structures $\big(K, G, (\gamma')_{\gamma\in\Gamma}\big)$ such that

1. $K$ is an algebraically closed field of the same characteristic as $E$,
2. $G$ is a subgroup of $K^\times$,
3. $\gamma \mapsto \gamma' \ : \Gamma \to G$ is a group homomorphism,
4. $\big(K, G, (\gamma')_{\gamma\in\Gamma}\big)$ satisfies the Mann axioms of $\Gamma$.

Here $\gamma'$ is the interpretation of (the name of) $\gamma$ in $\big(K, G, (\gamma')_{\gamma\in\Gamma}\big)$. If $E$ is algebraically closed, then $\big(E, \Gamma, (\gamma)_{\gamma\in\Gamma}\big)$ is clearly a model of $\mathrm{ACF}(\Gamma)$. The theory $\mathrm{ACF}(\Gamma)$ is never complete, but we can classify its models up to elementary equivalence.

**Theorem 5.6.** *Let* $\big(K, G, (\gamma)\big)$ *and* $\big(K', G', (\gamma)\big)$ *be models of* $\mathrm{ACF}(\Gamma)$. *Then* $\big(K, G, (\gamma)\big) \equiv \big(K', G', (\gamma)\big)$ *if and only if* $\big(G, (\gamma)\big) \equiv \big(G', (\gamma)\big)$ *as groups with distinguished elements.*