

THORN INDEPENDENCE IN THE FIELD OF REAL NUMBERS WITH A SMALL MULTIPLICATIVE GROUP

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ABSTRACT. We characterize \mathfrak{b} -independence in a variety of structures, focusing on the field of real numbers expanded by predicate defining a dense multiplicative subgroup, G , satisfying the Mann property and whose p th powers are of finite index in G . We also show such structures are super-rosy and eliminate imaginaries up to codes for small sets.

1. INTRODUCTION

We build on results of van den Dries and Günaydın in [3]. There the authors investigate the model theory of pairs (K, G) where K is either an algebraically closed field or a real closed field, and G is a multiplicative subgroup of K^\times with the Mann Property. While the definition of the Mann property is somewhat lengthy (and we postpone the precise definition to Section 5), roughly the Mann Property is a condition insuring that linear equations have few solutions in G . Among other things, the Mann property implies that G is *small* (in a technical sense defined below). Moreover, such groups are quite natural. Any group contained in the divisible hull of a finitely generated group, i.e. any finite rank group, has the Mann property.

In the case where K is real closed (henceforth we distinguish this case by referring to K as R), the additional hypothesis that G is a dense subgroup of $R^{>0}$ is used.

Among other results, van den Dries and Günaydın obtain good descriptions of the definable sets in both cases and a good description of dimension when K is algebraically closed, assuming G is ω -stable. In particular, the pair (K, G) is shown to be ω -stable of Morley rank ω .

We extend the results of [3] by obtaining a description of dimension for R real closed and G such that for each prime number, p , the subgroup of G consisting of p th-powers has finite index in G . To do this, we need to refine slightly the description of definable sets, focusing on a certain collection of definable sets we call “basic small”, and introduce the notion of \mathfrak{b} -rank. In particular, we prove that the pair (R, G) is super-rosy of \mathfrak{b} -rank ω . We then use this fact to obtain some partial results about elimination of imaginaries.

Now we state these results precisely.

Theorem 1.1. *Let R be a real closed field and G a dense subgroup of $R^{>0}$ with the Mann property and such that for each prime number, p , the subgroup of G consisting*

Date: June 1st, 2007.

1991 Mathematics Subject Classification. Primary 03C10, 03C45, 03C64.

Key words and phrases. rosy theories, Mann property, dense pairs, \mathfrak{o} -minimal theories.

The authors would like to thank Lou van den Dries for his close reading and helpful comments.

of p th-powers in G has finite index in G . Then in the language of ordered rings augmented with a unary predicate for G , we have

- (1) G has p -rank 1, and
- (2) (R, G) has p -rank ω .

Hence, (R, G) is super-rosy.

Theorem 1.2. *Let (R, G) be as in the previous theorem. Enlarge (R, G) by adding sufficiently many sorts of $(R, G)^{\text{eq}}$ so that the resulting structure has a code for every basic small subset of R^k , for each k . Then this structure eliminates imaginaries.*

While our primary interest is in subgroups of \mathbb{R} with the Mann property, we obtain Theorems 1.1 and 1.2 as applications of a more general result:

Theorem 1.3. *Suppose that $(R, +, \dots)$ is an o -minimal expansion of a group in the language \mathcal{L} . Consider the expansion $\mathfrak{R} = (R, \mathcal{G}, +, \dots)$ in the language $\mathcal{L}_{\mathcal{G}} = \mathcal{L} \cup \{\mathcal{G}\}$ where \mathcal{G} is a unary predicate. Suppose that for each $\mathfrak{R}' = (R', \dots)$ with $\mathfrak{R}' \equiv \mathfrak{R}$:*

- (1) $\mathcal{G}(R')$ is small, and contained in some interval, $(a, \infty) \subseteq R'$, in which it is dense.
- (2) Each $\mathcal{L}_{\mathcal{G}}$ -formula $\psi(x)$ is equivalent to a boolean combination¹ of formulas of the form $\exists \vec{y}(\mathcal{G}(y_1) \wedge \dots \wedge \mathcal{G}(y_j) \wedge \varphi(x, \vec{y}))$ where φ is an \mathcal{L} -formula.
- (3) For each tuple \vec{a} from R' and $\mathbb{D} \subseteq \mathcal{G}(R')^n$, definable over \vec{a} , there are an \mathcal{L} -definable set \mathbb{E} , and a definable \mathbb{S} , which is a dense subset of $\mathcal{G}(R')^n$, with \mathbb{E} and \mathbb{S} over \vec{a} , such that $\mathbb{D} = \mathbb{E} \cap \mathbb{S}$. Furthermore, when $n = 1$, \mathbb{D} can be written as a finite union of such $\mathbb{E} \cap \mathbb{S}$, where \mathbb{S} is, in addition, \emptyset -definable.

Then \mathfrak{R} is super-rosy of p -rank less than or equal to ω and p -rank of $\mathcal{G}(R)$ is 1. Moreover, if \mathfrak{R} includes a field structure, the p -rank of \mathfrak{R} equals ω .

For the definition of *small*, see 1.15.

The reader will note that if conditions (1) and (2) hold in a given model, they hold in any elementarily equivalent model, and if condition (3) holds in a sufficiently saturated model, it holds in any elementarily equivalent model. The reader will further note that condition (3) above seems quite technical. In many cases, a much more natural (and stronger) condition holds. Namely,

(3)' For each definable $\mathbb{D} \subseteq \mathcal{G}(R)^k$ there is an \mathcal{L} -definable set \mathbb{E} such that $\mathbb{D} = \mathbb{E} \cap \mathcal{G}(R)^k$.

However, in cases that are of particular interest to us, such as $\mathfrak{R} = (\mathbb{R}, \mathcal{G}, +, \cdot)$ and $\mathcal{G}(R) = 2^{\mathbb{Z}}3^{\mathbb{Z}}$, (3)' fails. To understand why (3) is not as unnatural as it may first appear, the reader may skip ahead to Section 5.

Theorem 1.4. *Let \mathfrak{R} be as in the previous theorem. Enlarge \mathfrak{R} by adding sufficiently many sorts of \mathfrak{R}^{eq} so that the resulting structure has a code for every basic small subset of R^k . Assume in addition, given any set of parameters A , and any interval I defined over A , that $\text{scl}(A) \cap I$ is not contained in any small set (see 1.15 and 2.12 for the appropriate definitions). Then this structure eliminates imaginaries.*

¹Throughout the paper, we use “boolean combination of . . .” to mean “an element of the ambient boolean algebra generated by . . .”.

In addition to applying to structures satisfying the conditions of Theorem 1.1, Theorems 1.3 and 1.4 also apply to the structures studied in [2], namely dense pairs of o-minimal expansions of ordered abelian groups. Also we note that we answer the question of Miller and Speissegger from [7] of whether $(\mathbb{R}, 2^{\mathbb{Z}}3^{\mathbb{Z}})$ has o-minimal open core. (See the end of Section 3 for definitions and the main part of the proof, and Section 5 for its application to expansions of the reals by groups with the Mann Property.)

Conventions and Notation. An \mathcal{L} -structure, e.g. $\mathfrak{A} = (R, +^{\mathfrak{A}}, \cdot^{\mathfrak{A}}, <^{\mathfrak{A}}, 0^{\mathfrak{A}}, 1^{\mathfrak{A}})$, consists of an underlying set, e.g. R , together with an interpretation of each symbol from the language, e.g. $+^{\mathfrak{A}}, \cdot^{\mathfrak{A}}, <^{\mathfrak{A}}, 0^{\mathfrak{A}}, 1^{\mathfrak{A}}$. We drop the superscripts when no confusion results. Capital letters in the Fraktur font, e.g. \mathfrak{M} and \mathfrak{A} , indicate structures. The universes of these structures are denoted by the corresponding capital letters in the normal font. For instance, M and R are the respective universes of the structures above.

We use the letters x, y, z, w as variables, and the letters a, b, c , etc., to indicate elements of the universe of a structure. We distinguish between elements from M and tuples from M^n by using vector notation for tuples. For example, \vec{x}, \vec{y} and \vec{a}, \vec{b} as opposed to x, y and a, b .

We use φ, ψ , and θ to indicate formulas. When no confusion results, we suppress the parameters, writing, for instance, $\varphi(\vec{x})$ even when the formula is not over the empty set. Likewise, when we say definable, we mean definable with parameters.

To save ourselves from constantly worrying about the length of our tuples, when \vec{x} is an n -tuple, we write M^n as $M_{\vec{x}}$. The set defined by a formula $\varphi(\vec{x})$ is denoted by $\varphi(M_{\vec{x}})$.

We use capital letters in blackboard bold to indicate definable sets, e.g. \mathbb{D}, \mathbb{E} , with the exceptions of \mathbb{N}, \mathbb{Q} , and \mathbb{R} , which are the sets of natural numbers, rational numbers, and real numbers, respectively. We denote the complement of \mathbb{D} as \mathbb{D}^c . We use f , possibly with subscripts, for definable functions. Also α, β will always indicate ordinals, m, n will always indicate natural numbers, and p will always indicate a prime number.

If we wish to emphasize that a definable set is defined with parameters, we write the parameters as a subscript. For example, suppose $\psi(\vec{y})$ defines \mathbb{E} and $\varphi(\vec{x})$ defines \mathbb{D} , where we have suppressed the parameters in both ψ and φ . If we then wish to emphasize that φ uses a parameter $\vec{e} \in M_{\vec{y}}$, we write $\mathbb{D}_{\vec{e}}$. For instance, we write

$$\exists \vec{y}(\psi(\vec{y}) \wedge \varphi(M_{\vec{x}}, \vec{y}))$$

as

$$\bigcup_{\vec{e} \in \mathbb{E}} \mathbb{D}_{\vec{e}}.$$

For a set C , we denote by $\mathcal{P}(C)$ the power set of C .

Definitions and Preliminaries. Now we introduce some definitions that we use in the remainder of the paper, together with some propositions from other papers which we also use.

Definition 1.5. Fix a theory, T , and a sufficiently saturated model $\mathfrak{M} \models T$. We work in \mathfrak{M}^{eq} . Let $\varphi(\vec{x}, \vec{y})$ be a formula without parameters, let $\vec{b} \in M_{\vec{y}}^{\text{eq}}$, and let C be a set of size less than the degree of saturation of \mathfrak{M} .

For $k \in \mathbb{N}$, the formula $\varphi(\vec{x}, \vec{b})$ is said to *k-b-divide* over C if there is $D \supseteq C$ such that $\text{tp}(\vec{b}/D)$ is not algebraic and the set of formulas $\{\varphi(\vec{x}, \vec{b}') \mid \vec{b}' \models \text{tp}(\vec{b}/D)\}$ is k -inconsistent. The formula is said to *b-divide* over C if it k -b-divides for some k .

The partial type $\pi(\vec{x}, \vec{b})$ is said to *b-fork* over C if it implies a disjunction of formulas (with arbitrary parameters), each of which b-divides over C .

We have defined what it means for a formula to b-divide over a set C . Sometimes, when the particulars of C are not important, we will simply say that a formula b-divides.

Remark 1.6. *By compactness, if φ k -b-divides, there is always a single formula $\theta(\vec{y}, \vec{d}) \in \text{tp}(\vec{b}/D)$ such that the set of formulas $\{\varphi(\vec{x}, \vec{b}') \mid \mathfrak{M} \models \theta(\vec{b}', \vec{d})\}$ is k -inconsistent.*

Also by compactness, if $\pi(\vec{x}, \vec{b})$ implies a disjunction of formulas that b-divide, π implies a finite disjunction of such formulas.

Definition 1.7. Let $A, B, C \subset M$ be smaller than the degree of saturation of \mathfrak{M} . Then \downarrow^b is defined as follows: $A \downarrow_C^b B$ if and only if $\text{tp}(\vec{a}/BC)$ does not b-fork over C for any tuple \vec{a} from A . If $A \downarrow_C^b B$ we say that *A is b-independent from B over C*. If it is clear from context, we will often just say *independent* for b-independent.

Definition 1.8. A theory T such that \downarrow^b is symmetric for T is called *rosy*.

Alternatively, rosiness could be defined in terms of local b-ranks being finite. However, we will not have need of any local ranks as the situation in which we find ourselves allows for a global b-rank, as defined below.

When working with an independence relation, we can define its foundation rank. For b-independence we have:

Definition 1.9. Let $p(x) \in S(A)$. For α an ordinal, we define $U^b(p) \geq \alpha$ inductively on α .

- (1) $U^b(p(x)) \geq 0$.
- (2) If $\alpha = \beta + 1$, we define $U^b(p(x)) \geq \alpha$ if there is a tuple a and a type $q(x, y)$ over A such that $q(x, a) \supset p(x)$, $U^b(q(x, a)) \geq \beta$ and $q(x, a)$ b-forks over A .
- (3) If α is a limit ordinal, then $U^b(p(x)) \geq \alpha$ if $U^b(p(x)) \geq \beta$ for all $\beta < \alpha$.

Remark 1.10. *It is perhaps worth noting that in a theory that is not rosy, b-forking may still be symmetric if one restricts the sorts that one considers. If thorn independence satisfies symmetry when restricted to the real sorts, one calls the theory real-rosy. For instance, the theory of algebraically closed valued fields is not a rosy theory, but b-forking, restricted to the field, residue field, and value group sorts, is an independence relation. Thus ACVF is real-rosy [4].*

Definition 1.11. b-rank is the least function taking values in $\text{On} \cup \{\infty\}$ satisfying the following:

- (1) $\text{b-rank}(\varphi(\vec{x}, \vec{b})) \geq 0$ if $\varphi(\vec{x}, \vec{b})$ is consistent.
- (2) $\text{b-rank}(\varphi(\vec{x}, \vec{b})) \geq \alpha + 1$ if there is $\psi(\vec{x}, \vec{c})$ that b-divides over \vec{b} , such that $\psi(\vec{x}, \vec{c}) \vdash \varphi(\vec{x}, \vec{b})$ and $\text{b-rank}(\psi(\vec{x}, \vec{c})) \geq \alpha$.
- (3) For λ a limit ordinal, $\text{b-rank}(\varphi(\vec{x}, \vec{b})) \geq \lambda$ if $\text{b-rank}(\varphi(\vec{x}, \vec{b})) \geq \alpha$ for all $\alpha < \lambda$.

The relation between \mathfrak{p} -rank and $U^{\mathfrak{p}}$ -rank is given by the following ([4]):

Fact 1.12. *For any type, p , $U^{\mathfrak{p}}(p) \leq \min\{\mathfrak{p}\text{-rank}(\varphi) \mid \varphi \in p\}$.*

In analogy with simple and stable theories, we make the following definition (which could be equivalently stated in terms of $U^{\mathfrak{p}}$ -rank, see [4]):

Definition 1.13. A complete theory is *super-rosy* if every formula has ordinal \mathfrak{p} -rank.

The corollary of the Coordinatization Theorem of [8] stated below will simplify our proof of super-rosiness:

Corollary 1.14. *Given a complete theory T , if every formula in one free variable $\varphi(x, \bar{b})$ has ordinal \mathfrak{p} -rank, then T is super-rosy.*

Definition 1.15. Let $\mathfrak{M} := (M, \dots)$ be an ordered structure. A definable set $\mathbb{D} \subset M^k$ is *large* iff there is some m , an interval $I \subseteq M$ and a function $f : \mathbb{D}^m \rightarrow I$.

A definable set \mathbb{S} is *small* iff it is not large.

Note that this definition of small differs from the conventions of [3]. There the adjective “small” also applies to sets that are not definable, but does not apply to subsets of M^n for $n > 1$. In addition, in [3], the notion of small set is defined for arbitrary, possibly unordered, structures. One of the cases we wish to consider, however, is dense pairs of ordered abelian groups. In this setting, a bounded interval would be small under the definition of [3]. Our definition for small, when restricted to definable subsets of a model (R, G) satisfying the hypotheses of Theorem 1.3 will turn out to be G -small, as defined in [2]. When R in addition has a field structure all three definitions will coincide (for definable subsets of R).

Fact 1.16. *Let \mathfrak{M} be an o -minimal structure. Let $\{\varphi(M, \bar{a})\}_{\bar{a} \in \mathbb{A}}$ be a definable family of subsets of M , each of which by o -minimality may be decomposed into a finite union of points and open intervals. Then the minimal number of points and the minimal number of open intervals in any such decomposition are definable properties of \bar{a} .*

Unless stated otherwise, \mathcal{L} denotes a language extending the language of ordered abelian groups, \mathcal{G} a unary predicate not in \mathcal{L} , $\mathfrak{R} = (R, G)$ denotes a structure satisfying the conditions of Theorem 1.3, although one may think of \mathfrak{R} as a structure satisfying the conditions of Theorem 1.1. Following our normal conventions, we should refer to the set defined by $\mathcal{G}(x)$ as \mathbb{G} , but we simply write it as G . We use $\mathfrak{R}|_{\mathcal{L}}$ to denote the reduct of \mathfrak{R} to \mathcal{L} .

2. SMALL SETS

We first make a definition and a technical observation.

Definition 2.1. A k -valued function, $F : A \xrightarrow{k} B$ is a function from A to $\{S \in \mathcal{P}(B) : |S| \leq k\}$. The *graph* of such an F is $\{(a, b) \in A \times B : b \in F(a)\}$, and its *image* is $\{b \in B : b \in F(a) \text{ for some } a \in A\}$. If $F : \mathbb{D} \rightarrow \mathbb{E}$ where $\mathbb{D} \subseteq R^m, \mathbb{E} \subseteq R^n$, then we say F is *definable in \mathfrak{R}* , if its graph is.

We define the composition of such functions as follows:

Definition 2.2. Consider $F_1 : A \xrightarrow{k_1} B$ and $F_2 : B \xrightarrow{k_2} C$. We define $F_2 \circ F_1 : A \xrightarrow{k_3} C$ by setting $F_2 \circ F_1(a) := \{c : \exists b \in F_1(a) \text{ and } c \in F_2(b)\}$, where $k_3 := k_1 \cdot k_2$.

Lemma 2.3. *Let $\mathfrak{M} = (M, <, \dots)$ be any ordered structure, \mathbb{E}, \mathbb{F} be definable subsets of M^m, M^n respectively, and $F : \mathbb{E} \xrightarrow{k} \mathbb{F}$ be a k -valued function. Then there is a function $f : \mathbb{E}^k \rightarrow \mathbb{F}$ with the same image as F . If there are two definable elements of \mathbb{E} then f has the same parameters as F .*

Proof. Pick distinct a_1, a_2 definable elements contained in \mathbb{E} (adding parameters if necessary). Suppose that $e \in \mathbb{E}$ is not equal to a_1 . Set $f((e, a_1, \dots, a_1))$ to be the least element of $F(e)$, set $f((a_1, e, a_1, \dots, a_1))$ to be the second least element of $F(e)$, etc. Now suppose that $e = a_1$. Set $f((e, a_2, \dots, a_2))$ to be the least element of $F(e)$, etc. Finally, for any $\vec{e} \in \mathbb{E}^k$ on which f is not yet defined, set $f(\vec{e})$ equal to the least element of $F(a_1)$. \square

Let us make a couple of observations about the notion of small as it applies in the setting of groups. Let $(M, +, \dots)$ be an expansion of a group. Then the complement of any small set, \mathbb{S} , is large. This can be seen, for instance, by considering the map $f : M^2 \rightarrow M$ given by $(m_1, m_2) \mapsto m_1 + m_2$. Suppose some element, $m_0 \in M$, is not in the image of $(\mathbb{S}^c)^2$ under f . Then

$$m_0 \in \bigcap_{m \notin \mathbb{S}} \mathbb{S} + m.$$

Thus, $m_0 - \mathbb{S}$ contains \mathbb{S}^c . Now the 2-valued function $\mathbb{S} \xrightarrow{2} M, s \mapsto \{s, m_0 - s\}$ witnesses that \mathbb{S} is large, which is a contradiction. Actually we need a stronger statement:

Lemma 2.4. *Let $(M, +, <, \dots)$ be an expansion of an ordered group, and $I = (a, b) \subseteq M$ be a nonempty interval, and $\mathbb{S} \subseteq M$ a small set. Then $I \setminus \mathbb{S}$ is large.*

Proof. Let $f : M^2 \rightarrow M$ be defined as in the previous paragraph. Let $J = (a+b, 2b)$. We show that $f((I \setminus \mathbb{S})^2) \supseteq J$. For a contradiction, let $m_0 \in J \setminus f((I \setminus \mathbb{S})^2)$. Then, reasoning as above, $-(\mathbb{S} \cup I^c) + m_0 \supseteq I \setminus \mathbb{S}$. Noting that

$$I^c + m_0 = (-\infty, -b + m_0) \cup (-a + m_0, \infty),$$

we see that this yields $-\mathbb{S} + m_0 \supseteq (-b + m_0, b)$, contradicting the smallness of \mathbb{S} . \square

Definition 2.5. We say a definable set \mathbb{D} is *small in an interval I* if $\mathbb{D} \cap I$ is small. We say a definable set \mathbb{D} is *cosmall in an interval I* if $\mathbb{D}^c \cap I$ is small.

Here we return from considering arbitrary ordered groups to the setting of Theorem 1.3.

Definition 2.6. A definable set \mathbb{X} is *basic* if it is defined by a formula of the form $\exists \vec{y}(\mathcal{G}(\vec{y}) \wedge \varphi(\vec{x}, \vec{y}))$ where $\varphi(\vec{x}, \vec{y})$ is a formula in \mathcal{L} , and by $\mathcal{G}(\vec{y})$, we mean $\mathcal{G}(y_1) \wedge \dots \wedge \mathcal{G}(y_n)$. Furthermore, we will refer to formulas of the form $\exists \vec{y}(\mathcal{G}(\vec{y}) \wedge \varphi(\vec{x}, \vec{y}))$ as *basic formulas*.

Remark 2.7. *Note that a set is basic if and only if it can be written as*

$$\bigcup_{\vec{g} \in G^n} \varphi(R_{\vec{x}}, \vec{g}).$$

where φ is an \mathcal{L} -formula. Note also that finite unions and intersections of basic sets are again basic. In particular, an interval intersect a basic set is again a basic set.

For our purposes the above characterization of definable sets is not quite sufficient; we obtain a more detailed description in the case of definable subsets of R (as opposed to R^n).

First we need to prove that if f_1 and f_2 are functions $R^n \rightarrow R$ definable in \mathcal{L} then $\bigcup_{\vec{g} \in G^n} (f_1(\vec{g}), f_2(\vec{g}))$ is a finite union of intervals. This is clear when f_1 and f_2 are functions in one variable. In general, it is slightly less clear. However, it is a consequence of the cell decomposition theorem for o-minimal structures and the following two lemmas.

The first of the two lemmas shows that subsets of G^k are in a sense well approximated by \mathcal{L} -definable sets. We already know that for any such set, \mathbb{D} , there is an \mathcal{L} -definable set \mathbb{E} such that \mathbb{D} is dense in $\mathbb{E} \cap G^k$. It is not the case that \mathbb{D} will necessarily be dense in \mathbb{E} . For instance, let $(R, G) := (\mathbb{R}, 2^{\mathbb{Q}})$. Consider the plane, $\mathbb{P} \subset \mathbb{R}^3$ defined by $z - 3y = 0$. Let $\mathbb{D} := \mathbb{P} \cap G^3$. Then \mathbb{D} is just the copy of G lying on the x -axis, and not dense in \mathbb{P} . Clearly, in this example, had we chosen \mathbb{E} as the x -axis, rather than the plane \mathbb{P} we would have obtained the density we desired. We prove that in general, choosing \mathbb{E} carefully, we can in fact obtain density in \mathbb{E} .

Lemma 2.8. *For any $\mathbb{D} \subseteq G^n$, there is \mathcal{L} -definable \mathbb{B} such that \mathbb{D} is a dense subset of \mathbb{B} . Moreover \mathbb{B} is defined over the same parameters as \mathbb{D} .*

Proof. Let \mathbb{D} be definable over \vec{a} . By the hypotheses of Theorem 1.3, we know that there are an \vec{a} -definable \mathbb{E} and \mathbb{S} such that \mathbb{E} is \mathcal{L} -definable, \mathbb{S} is a dense subset of G^n , and $\mathbb{D} = \mathbb{E} \cap \mathbb{S}$. We proceed by induction on the dimension, k , of \mathbb{E} to find an $\mathbb{B} \subseteq \mathbb{E}$, \mathcal{L} -definable over \vec{a} with \mathbb{D} a dense subset of \mathbb{B} . There is nothing to prove for $k = 0$.

Now suppose we have proven the claim for $j < k$. We may assume that \mathbb{E} is a cell: write \mathbb{E} as $\mathbb{E}_1 \cup \dots \cup \mathbb{E}_l$, with each \mathbb{E}_i a cell defined over \vec{a} . If \mathbb{E}_i is of dimension less than k , then we may apply the inductive hypothesis to $\mathbb{E}_i \cap \mathbb{S}$. Thus we may assume \mathbb{E} is a cell of dimension k .

As \mathbb{E} is a cell, we may choose a projection $\pi : \mathbb{E} \rightarrow \pi(\mathbb{E}) \subseteq R^k$ so that π is a homeomorphism. Now choose an \vec{a} -definable \mathbb{E}' and \mathbb{S}' such that \mathbb{E}' is \mathcal{L} -definable, \mathbb{S}' is a dense subset of G^k , and $\pi(\mathbb{D}) = \mathbb{E}' \cap \mathbb{S}'$. Again, we may divide \mathbb{E}' into cells, say $\mathbb{E}'_1 \cup \dots \cup \mathbb{E}'_m$. For each i , either \mathbb{E}'_i has dimension k , in which case it is open and $\pi(\mathbb{D}) \cap \mathbb{E}'_i = \mathbb{E}'_i \cap \mathbb{S}'$ is dense in \mathbb{E}'_i or \mathbb{E}'_i has dimension less than k and we may apply induction to assume $\pi(\mathbb{D}) \cap \mathbb{E}'_i$ is a dense subset of \mathbb{E}'_i . Thus $\pi(\mathbb{D})$ is a dense subset of \mathbb{E}' .

Now let $\mathbb{B} := \pi^{-1}(\mathbb{E}')$. As π is a homeomorphism, \mathbb{D} is a dense subset of \mathbb{B} and since π is \mathcal{L} -definable, so is \mathbb{B} . We observe that \mathbb{B} is definable over \vec{a} . \square

The second of the two lemmas presents a condition under which a set definable in (R, G) is actually an interval.

Lemma 2.9. *Let $\mathbb{B} \subseteq R^n$ be a cell such that f_1 and f_2 are continuous on \mathbb{B} , $\mathbb{B} \cap G^n$ is dense in \mathbb{B} , and $f_1(\vec{x}) < f_2(\vec{x})$. Then $\bigcup_{\vec{g} \in \mathbb{B} \cap G^n} (f_1(\vec{g}), f_2(\vec{g}))$ is an interval.*

Proof. Let $a = \inf f_1(\mathbb{B})$ and $b = \sup f_2(\mathbb{B})$. Let $d \in (a, b)$; we wish to show that $d \in \bigcup_{\vec{g} \in \mathbb{B} \cap G^n} (f_1(\vec{g}), f_2(\vec{g}))$. For some $c_1 \in \mathbb{B}$, $f_1(c_1) < d$. Clearly if $f_2(c_1) > d$, we are done, so we may assume that $f_2(c_1) < d$. Likewise we may assume that there is some c_2 such that $d < f_1(c_2) < f_2(c_2)$. Note that $(f_1 + f_2)(c_1) < 2d$ while $(f_1 + f_2)(c_2) > 2d$. Thus, by the continuity of f_1 and f_2 , and by the connectedness of \mathbb{B} , there is c_3 such that $(f_1 + f_2)(c_3) = 2d$. Since $f_1 < f_2$, we conclude that

$d \in (f_1(c_3), f_2(c_3))$. By the density of $G^n \cap \mathbb{B}$ in \mathbb{B} we may find $\vec{g} \in \mathbb{B} \cap G^n$ such that $d \in (f_1(\vec{g}), f_2(\vec{g}))$. \square

Corollary 2.10. *If f_1 and f_2 are functions $R^n \rightarrow R$ which are definable in \mathcal{L} , then $\bigcup_{\vec{g} \in G^n} (f_1(\vec{g}), f_2(\vec{g}))$ is a finite union of intervals.*

Proof. Recall that \mathfrak{A} restricted to \mathcal{L} is an o-minimal structure. Given f_1 and f_2 , \mathcal{L} -definable n -ary functions, we can decompose R^n as a finite union of disjoint cells, \mathbb{C}_i , where on each \mathbb{C}_i , f_1, f_2 are continuous, and either the functions coincide on every point of \mathbb{C}_i or else one of the functions is strictly larger on every point of \mathbb{C}_i . By Lemma 2.8, we may shrink each \mathbb{C}_i until we obtain a cell, \mathbb{B}_i , such that $\mathbb{B}_i \cap G^n$ is a dense subset of \mathbb{B}_i . By Lemma 2.9, on each such cell, $\bigcup_{\vec{g} \in \mathbb{B}_i \cap G^n} (f_1(\vec{g}), f_2(\vec{g}))$ is an interval. \square

Proposition 2.11. *Let $\mathbb{D} \subseteq R$ be definable in \mathfrak{A} . Then there is a finite partition $-\infty = a_0 < a_1 < \dots < a_m = \infty$ of R such that \mathbb{D} is either small or cosmall in (a_{i-1}, a_i) for $i = 1, \dots, m$. Furthermore, if \mathbb{D} is definable from \vec{d} , so is the partition $-\infty = a_0 < a_1 < \dots < a_m = \infty$.*

Proof. We first assume that \mathbb{D} is basic. So $\mathbb{D} = \bigcup_{\vec{g} \in G^n} \varphi(R, \vec{g})$, where $\varphi(x, \vec{g})$ is an \mathcal{L} -formula. By the o-minimality of $\mathfrak{A}|_{\mathcal{L}}$, each $\varphi(x, \vec{g})$ defines a finite union of points and intervals, and there is a uniform bound on the number of these points and intervals. By Fact 1.16, we may assume without loss of generality that each $\varphi(x, \vec{g})$ defines either a single point or a single interval.

First let us consider the case where $\varphi(x, \vec{g})$ is a single point. As there is a definable surjection from G^n onto \mathbb{D} , we see that \mathbb{D} is small.

Now we consider the case where each $\varphi(x, \vec{g})$ is an interval. There are \mathcal{L} -definable $f_1, f_2: R^n \rightarrow R$ such that $\varphi(R, \vec{g}) = (f_1(\vec{g}), f_2(\vec{g}))$. By Corollary 2.10,

$$\bigcup_{\vec{g} \in G^n} (f_1(\vec{g}), f_2(\vec{g}))$$

is a finite union of intervals. By o-minimality, the endpoints of these intervals are definable over any parameters from which the finite union of intervals may be defined.

Thus, we have our result if $\mathbb{D} = \bigcup_{\vec{g} \in G^n} \varphi(R, \vec{g})$.

Now assume \mathbb{D} and \mathbb{E} satisfy the conclusion. To complete the proof, we must show that \mathbb{D}^c and $\mathbb{D} \cup \mathbb{E}$ also have the desired property. But this is clear. \square

Definition 2.12. We say that \vec{e} is in the *small closure* of A iff \vec{e} is contained in a small set defined with parameters from A . We denote the small closure of A by $\text{scl}(A)$.

Definition 2.13. We say that a set, $\mathbb{S} \subseteq R^k$, is G -bound iff there is an \mathcal{L} -definable $f: R^n \rightarrow R^k$ such that $\mathbb{S} \subseteq f(G^n)$.

It is clear that G -bound implies small. We proceed to prove the converse.

Lemma 2.14. *Any basic small set \mathbb{S} is G -bound. Furthermore, assuming that there are two definable elements of R , the function f witnessing that \mathbb{S} is G -bound is definable over the same parameters as \mathbb{S} .*

Proof. Note that if $\mathbb{S} \subset R^k$ is a basic small set, so is each projection of \mathbb{S} to R ; and that cartesian products of G -bound sets are G -bound. Thus it suffices to consider small subsets of R .

Suppose that \mathbb{S} is defined with parameters \vec{a} . Let \mathbb{S} be

$$\bigcup_{\vec{g} \in G^n} \varphi(R, \vec{g}, \vec{a})$$

where $\varphi(x, \vec{y}, \vec{z})$ is a parameter-free \mathcal{L} -formula. Since $\mathfrak{A}|_{\mathcal{L}}$ is o-minimal, each set $\varphi(R, \vec{g}, \vec{a})$ is a finite collection of points and intervals. It is easy to see that any set containing an open interval is large, so each $\varphi(R, \vec{g}, \vec{a})$ is a finite set. By o-minimality, there is a uniform bound k to the size of $\varphi(R, \vec{g}, \vec{a})$ for each $\vec{g} \in G^n$.

Thus mapping \vec{g} to $\varphi(R, \vec{g}, \vec{a})$ gives us a k -valued (and \vec{a} -definable) function, F , in the language \mathcal{L} such that $F(G^n) = \mathbb{S}$. By 2.3, we may replace this with an actual function, f . (Although if 0 is the only definable element of R , we may have to add an additional parameter in R .) \square

Remark 2.15. *Note that even when 0 is the only definable element, \mathbb{S} is still the image G under a k -valued function which is definable with the same parameters as \mathbb{S} .*

Lemma 2.16. *Let $\varphi(x, \vec{d})$ define \mathbb{D} . Then there are a partition $-\infty = a_0 < \dots < a_n = \infty$ and basic small sets $\mathbb{S}_1, \dots, \mathbb{S}_n$ such that $\mathbb{D} \cap [a_{i-1}, a_i]$ either is contained in \mathbb{S}_i , or contains $\mathbb{S}_i^c \cap [a_{i-1}, a_i]$. Furthermore, the partition and each \mathbb{S}_i are definable over \vec{d} .*

Proof. Note that $\varphi(x, \vec{d})$ is equivalent to a boolean combination of basic formulas. We proceed by induction, using repeatedly that the intersection of a basic set with an interval is again a basic set.

Suppose that $\varphi(x, \vec{d})$ is a basic formula. By Proposition 2.11, there is a \vec{d} -definable partition $-\infty = a_0 < \dots < a_n = \infty$ such that $\mathbb{D} \cap [a_{i-1}, a_i]$ either is small or cosmall. If $\mathbb{D} \cap [a_{i-1}, a_i]$ is small, let $\mathbb{S}_i := \mathbb{D} \cap [a_{i-1}, a_i]$. If $\mathbb{D} \cap [a_{i-1}, a_i]$ is cosmall in $[a_{i-1}, a_i]$, then $\mathbb{D} \cap [a_{i-1}, a_i]$ is a finite union of intervals, by Lemma 2.10. Thus, since it is small, $[a_{i-1}, a_i] \setminus \mathbb{D}$ is a finite collection of points. Let \mathbb{S}_i be this finite collection of points. Note that in either case, by Proposition 2.11, \mathbb{S}_i can be defined over \vec{d} .

Now suppose that $\varphi = \varphi_1 \wedge \varphi_2$. Let $\mathbb{E}_1 := \varphi_1(R, \vec{d})$ and let $\mathbb{E}_2 := \varphi_2(R, \vec{d})$. By induction, there are a partition $-\infty = b_0 < \dots < b_m = \infty$ and basic small sets $\tilde{\mathbb{S}}_1, \dots, \tilde{\mathbb{S}}_m$ with the desired property with respect to \mathbb{E}_1 . Likewise there are a partition $-\infty = c_0 < \dots < c_n = \infty$ and basic small sets $\tilde{\mathbb{S}}_{m+1}, \dots, \tilde{\mathbb{S}}_{m+n}$ with the desired property with respect to \mathbb{E}_2 . Let $-\infty = a_0 < \dots < a_l = \infty$ be the union of these two partitions. Then $\mathbb{D} \cap [a_{i-1}, a_i]$ is either small or cosmall.

If $\mathbb{D} \cap [a_{i-1}, a_i]$ is small, then either \mathbb{E}_1 or \mathbb{E}_2 is small in $[a_{i-1}, a_i]$. Without loss of generality, we may assume it is \mathbb{E}_1 . Note that $[a_{i-1}, a_i]$ is contained in $[b_{k-1}, b_k]$ for some k . Let $\mathbb{S}_i := \tilde{\mathbb{S}}_k \cap [a_{i-1}, a_i]$. As $\tilde{\mathbb{S}}_k$ is \vec{d} -definable and contains $\mathbb{E}_1 \cap [b_{k-1}, b_k]$, we see that \mathbb{S}_i satisfies the desired properties.

If $\mathbb{D} \cap [a_{i-1}, a_i]$ is cosmall, then both \mathbb{E}_1 and \mathbb{E}_2 are cosmall in $[a_{i-1}, a_i]$. There are j, k , such that $[a_{i-1}, a_i] \subseteq [b_{j-1}, b_j]$ and $[a_{i-1}, a_i] \subseteq [c_{k-1}, c_k]$. Thus, $\mathbb{E}_1 \cap [a_{i-1}, a_i]$ contains $\tilde{\mathbb{S}}_j^c \cap [a_{i-1}, a_i]$, and $\mathbb{E}_2 \cap [a_{i-1}, a_i]$ contains $\tilde{\mathbb{S}}_{m+k}^c \cap [a_{i-1}, a_i]$. Thus, \mathbb{D} contains $(\tilde{\mathbb{S}}_j \cup \tilde{\mathbb{S}}_{m+k})^c \cap [a_{i-1}, a_i]$. We let $\mathbb{S}_i := (\tilde{\mathbb{S}}_j \cup \tilde{\mathbb{S}}_{m+k})^c \cap [a_{i-1}, a_i]$

Now suppose that $\varphi = \neg\varphi_0$. Let \mathbb{E} be defined by φ_0 . By induction there is a partition $-\infty = a_0 < \dots < a_n = \infty$ and basic small sets $\mathbb{S}_1, \dots, \mathbb{S}_n$ such that $\mathbb{E} \cap [a_{i-1}, a_i]$ either is contained in \mathbb{S}_i , or contains $\mathbb{S}_i^c \cap [a_{i-1}, a_i]$, and the \mathbb{S}_i are defined from \vec{d} . But this partition and these small sets work for \mathbb{D} as well. \square

From the previous two lemmas (as well as Lemma 2.4), we obtain the following two corollaries:

Corollary 2.17. *If \mathbb{S} is a small set, then it is contained in a basic small set and, hence, \mathbb{S} is G -bound.*

Proof. Let $\mathbb{S} \subset R^k$. Let π_i be the projection onto the i th coordinate. Let $\mathbb{S}_i := \pi_i(\mathbb{S})$. By Lemma 2.16, take $\tilde{\mathbb{S}}_i$, a basic small set containing \mathbb{S}_i . Then $\tilde{\mathbb{S}}_1 \times \dots \times \tilde{\mathbb{S}}_k$ is a basic small set containing \mathbb{S} . As \mathbb{S} is contained in a G -bound set, it is itself G -bound. \square

Corollary 2.18. *A tuple, \vec{c} , is in the small closure of A if and only if there is an \mathcal{L}_A -definable k -valued function, $F(\vec{x})$ and some $\vec{g} \in G^n$ such that $\vec{c} \in F(\vec{g})$. Thus, if $\vec{a} \in \text{scl}(\vec{b})$ and $\vec{b} \in \text{scl}(\vec{c})$ then $\vec{a} \in \text{scl}(\vec{c})$.*

Proof. If $\vec{c} \in \text{scl}(A)$ then there is a small set $\mathbb{S}_{\vec{c}}$ defined with parameters \vec{a} from A that contains \vec{c} . The set $\mathbb{S}_{\vec{c}}$ is contained in a basic small set, also defined over A , and this basic small set is the image of a k -valued function on G^n . Conversely, such a set is G -bound, and hence small. Moreover, if $\vec{a} \in \text{scl}(\vec{b})$ and $\vec{b} \in \text{scl}(\vec{c})$ then this is witnessed by k_1 and k_2 -valued functions, F_1 and F_2 respectively, with $F_1 = F_1(\vec{x}, \vec{b})$ and $F_2 = F_2(\vec{y}, \vec{c})$. Thus $F_3 := F_1(\vec{x}, F_2(\vec{y}, \vec{c}))$ witnesses that $\vec{a} \in \text{scl}(\vec{c})$. \square

In addition, we have the following corollary:

Corollary 2.19. *A finite union of small sets is again a small set.*

Proof. Finite unions of G -bound sets are again G -bound, by Lemma 2.2 of [3]. \square

While we rely on [3] for the above proof, we note that the corollary also follows as a special case of Proposition 2.22 below.

Remark 2.20. *Since scl is transitive, and $\text{scl}(\emptyset)$ is infinite, (and in particular, contains at least one non-zero element) we may add an element of $\text{scl}(\emptyset)$ to the language without affecting small closure. Thus we may assume that R contains at least two definable elements, and henceforth, we will assume that we may replace each k -valued function with an actual function.*

Remark 2.21. *Note that, unlike the algebraic closure of A , $\text{scl}(A)$ depends on the model containing A .*

Although the following proposition is not used in the proofs of this article's main theorems, it is interesting to note that a small definable union of small sets is again a small set.

Proposition 2.22. *If \mathbb{D} is small, and $\mathbb{E}_{\vec{d}}$ is small for each $\vec{d} \in \mathbb{D}$, then $\bigcup_{\vec{d} \in \mathbb{D}} \mathbb{E}_{\vec{d}}$ is also small.*

Proof. First note that by Corollary 2.17, for each $\vec{d} \in \mathbb{D}$ there is a basic small set containing $\mathbb{E}_{\vec{d}}$. By compactness, the formula defining the basic small set may be chosen uniformly in \vec{d} . Thus, we may reduce to the case where \mathbb{D} and each $\mathbb{E}_{\vec{d}}$ are basic small.

Assume that the formula $\theta(\vec{x}, \vec{d})$ defines $\mathbb{E}_{\vec{d}}$ for every $\vec{d} \in \mathbb{D}$. Then, since $\mathbb{E}_{\vec{d}}$ is a basic small set, there are $\psi(\vec{y}) \in \text{tp}(\vec{d})$ and $f(\vec{x}, \vec{y})$ such that whenever $\vec{d}' \models \psi(\vec{y})$, we have $f(\vec{x}, \vec{d}'): G^k \rightarrow \mathbb{E}_{\vec{d}'}$. Note that k , ψ , and f may depend on \vec{d} . However by compactness, there is a finite covering of \mathbb{D} with sets defined by $\psi_1(\vec{y}), \dots, \psi_n(\vec{y})$, together with associated k_1, \dots, k_n and f_1, \dots, f_n . By taking $k = \max\{k_1, \dots, k_n\}$, we see that there is a definable function

$$f(\vec{x}, \vec{y}): G^k \times \mathbb{D} \rightarrow \bigcup_{\vec{d} \in \mathbb{D}} \mathbb{E}_{\vec{d}}$$

such that for any $\vec{d} \in \mathbb{D}$, $f(\vec{x}, \vec{d}): G^k \rightarrow \mathbb{E}_{\vec{d}}$.

Now suppose that $g: G^n \rightarrow \mathbb{D}$ witnesses that \mathbb{D} is small. Then let $h: G^{k+n} \rightarrow \bigcup_{\vec{d} \in \mathbb{D}} \mathbb{E}_{\vec{d}}$ be defined as follows:

$$h(\vec{a}_1, \vec{a}_2) := f(\vec{a}_1, g(\vec{a}_2)).$$

So $\bigcup_{\vec{d} \in \mathbb{D}} \mathbb{E}_{\vec{d}}$ is G -bound, and hence small. \square

Definition 2.23. For a set C , a function from $\mathcal{P}(C)$ to $\mathcal{P}(C)$ is a *closure operator* iff for any $A, B \subseteq C$

- (1) $A \subseteq \text{cl}(A)$,
- (2) $A \subseteq B$ implies $\text{cl}(A) \subseteq \text{cl}(B)$,
- (3) $\text{cl}(\text{cl}(A)) = \text{cl}(A)$.

Furthermore, we say that a closure operator is *finitary* when (2) is strengthened to (2') $b \in \text{cl}(A)$ iff $b \in \text{cl}(A_0)$ for some finite $A_0 \subseteq A$.

If the closure also satisfies the Steinitz exchange property, then we say that the closure operator gives rise to a *pregeometry*.

It is clear that the small closure satisfies (1), is finitary, and, by Corollary 2.18, satisfies (3). Thus we have proven:

Proposition 2.24. *The small closure, scl is a finitary closure operator on subsets of R .*

3. SUPER-ROSINESS OF (R, G)

In this section we prove Theorem 1.3. To do this, we will need to use the following propositions from [4]. Throughout this section, we assume that (R, G) is κ -saturated, for $\kappa > 2^{|\mathcal{L}_G|}$

Proposition 3.1. *If \mathbb{D} has \mathfrak{p} -rank α and $f: \mathbb{D} \rightarrow \mathbb{E}$, then \mathbb{E} has \mathfrak{p} -rank less than or equal to α . Furthermore, if the fibers of f are finite, we have equality.*

Proposition 3.2. *If \mathbb{D} has \mathfrak{p} -rank α and \mathbb{E} has \mathfrak{p} -rank less than α , then $\mathfrak{p}\text{-rank}(\mathbb{D} \setminus \mathbb{E})$ is α .*

Proposition 3.3. *If \mathbb{D} has \mathfrak{p} -rank α then \mathbb{D}^n has \mathfrak{p} -rank at least αn , and equality holds if $\alpha = 1$.*

Now we begin to analyze \mathfrak{p} -dividing in (R, G) . In what follows, $\mathcal{L}_G^{\text{eq}}$ refers to the language of $(R, G)^{\text{eq}}$.

Lemma 3.4. *Let $\varphi(x, \vec{b}_0)$ be a formula in $\mathcal{L}_G^{\text{eq}}$ with x a variable in the real sort. If $\varphi(R, \vec{b}_0)$ is an infinite set definable in \mathcal{L} , then $\varphi(x, \vec{b}_0)$ does not \mathfrak{p} -divide over the empty set.*

Proof. It may be worth pointing out that merely because the set $\varphi(R, \vec{b}_0)$ is definable in \mathcal{L} , we may not assume that φ is an \mathcal{L} -formula. For instance, \vec{b}_0 may come from a sort that does not even exist in $(\mathfrak{R}|_{\mathcal{L}})^{\text{eq}}$.

Assume, for a contradiction, that $\varphi(x, \vec{b}_0)$ does \mathfrak{p} -divide over the empty set. That is, $\text{tp}(\vec{b}_0)$ is non-algebraic, and there is some $\theta(\vec{y}, \vec{c})$ and some $k \in \mathbb{N}$ such that whenever $\vec{b}_1, \dots, \vec{b}_k$ are distinct elements of $\theta(R_{\vec{y}}, \vec{c})$, we have that $\varphi(x, \vec{b}_1) \wedge \dots \wedge \varphi(x, \vec{b}_k)$ is inconsistent. Since φ defines an infinite \mathcal{L} -definable set, by the \mathfrak{o} -minimality of $\mathfrak{R}|_{\mathcal{L}}$, it defines a finite collection of points and open intervals.

First note that we may assume that for each $\vec{b} \models \theta(\vec{y}, \vec{c})$, it is the case that $\varphi(x, \vec{b})$ defines a single interval, modifying φ and θ if necessary. (It is possible that for some $\vec{b} \models \theta(\vec{y}, \vec{c})$, $\varphi(x, \vec{b})$ defines a finite collection of points. First we modify θ to rule out this possibility. Then we replace $\varphi(x, \vec{b})$ with a formula defining the least of the intervals in the finite collection of points and intervals composing $\varphi(R, \vec{b})$.)

Now we wish to reduce to the case where $k = 2$. We may assume that $\varphi(x, \vec{y})$ does not $(k - 1)$ - \mathfrak{p} -divide. Replace $\varphi(x, \vec{y})$ with

$$\tilde{\varphi}(x, \vec{y}_1, \dots, \vec{y}_{k-1}) := \bigwedge_{i < k} \varphi(x, \vec{y}_i)$$

and replace θ with

$$\tilde{\theta}(\vec{y}_1, \dots, \vec{y}_{k-1}) := \theta(y_1) \wedge \dots \wedge \theta(y_{k-1}) \wedge \bigwedge_{i < j < k} \vec{y}_i < \vec{y}_j.$$

Now $\tilde{\varphi}$ clearly 2- \mathfrak{p} -divides.

Now we would like to find a contradiction by considering the union of the sets defined by $\varphi(x, \vec{b})$ for $\vec{b} \models \theta$, intersecting with G , and noting that it violates (3) of our assumptions on \mathfrak{R} from Theorem 1.3. First note that since G is a dense subset of (a, ∞) , we can assume that $\varphi(\mathfrak{R}, b)$ is contained in the closure of G for each $b \models \theta$ (possibly after reflecting the whole family over a and modifying θ). However, there is still no immediate contradiction since $\bigcup_{\vec{b} \models \theta} \varphi(R, \vec{b}) \cap G$ might still be a finite union of intervals in G . We can modify $\varphi(x, \vec{b})$ once again to define the interval with half the length but the same center as $\varphi(x, \vec{b})$. Now, the union of these intersect G cannot be written as a finite union of intervals intersect a dense subset of G . \square

Now we have all the tools in place to begin our proof of Theorem 1.3.

Theorem 1.3. *$\mathfrak{R} = (R, G)$ is super-rosy of \mathfrak{p} -rank less than or equal to ω and \mathfrak{p} -rank of G is 1, Moreover, if \mathfrak{R} includes a field structure, \mathfrak{p} -rank of \mathfrak{R} equals ω .*

Proof. First we wish to show that the \mathfrak{p} -rank of G is 1. For a contradiction, suppose that some formula $\varphi(x, \vec{b})$ which defines an infinite subset of G \mathfrak{p} -divides over the empty set. Say that $k, \theta(\vec{y}, \vec{c})$ are such that $\bigwedge_{i \leq k} \varphi(x, \vec{b}_i)$ is inconsistent for any k distinct elements $\vec{b}_1, \dots, \vec{b}_k$ satisfying $\theta(\vec{y}, \vec{c})$.

Then, by (3) of the hypotheses of Theorem 1.3, $\varphi(R, \vec{b})$ is a finite union of sets, each of which is either a point or an interval intersect an \emptyset -definable dense subset of G . Without loss of generality, we may assume that for each $\vec{b}' \models \theta(\vec{y}, \vec{c})$, it is the case that $\varphi(x, \vec{b}')$ defines a single interval, $\psi_1(R, \vec{b}')$, intersect an \emptyset -definable dense subset of G . Which \emptyset -definable set may depend on the type of \vec{b}' , but one such set, $\psi_2(R)$, must occur for infinitely many \vec{b}' . Modifying θ if necessary, we may assume that for all $\vec{b}' \models \theta(\vec{y}, \vec{c})$, we have that $\varphi(x, \vec{b}')$ defines the same set as $\psi_1(x, \vec{b}') \wedge \psi_2(x)$.

Thus we have that $\{\psi_1(x, \vec{b}') \wedge \psi_2(x) : \vec{b}' \models \theta(\vec{y}, \vec{c})\}$ is k -inconsistent. But by Lemma 3.4, $\psi_1(x, \vec{b}')$ does not \mathfrak{p} -divide, and so we may find an infinite $B = \{b_i : b_i \models \text{tp}(\vec{b}/\vec{c}), i < \alpha\}$ such that $\bigcap_{\vec{b}_i \in B} \psi_1(R, \vec{b}_i)$ is nonempty and, hence, contains an open interval (d_1, d_2) . But since $\psi_2(x)$ is a dense subset of G ,

$$\bigcap_{\vec{b}_i \in B} \varphi(R, \vec{b}_i) \supseteq (d_1, d_2) \cap \psi_2(R) \neq \emptyset,$$

which is a contradiction.

Second, we wish to show that the \mathfrak{p} -rank of $x = x$ is no larger than ω . Suppose that $\varphi(x, \vec{b})$ k - \mathfrak{p} -divides over the empty set, where, again, \vec{b} may come from any sort in \mathfrak{R}^{eq} . We observe that it suffices to show that $\mathbb{D}_{\vec{b}} := \varphi(R, \vec{b})$ must be a small set, since any small set is G -bound, and thus we may apply Proposition 3.1 and Proposition 3.3 to conclude that any G -bound set has finite \mathfrak{p} -rank. Then we will have shown that any formula, $\varphi(x, \vec{b})$, which \mathfrak{p} -divides has finite \mathfrak{p} -rank, and, thus, $\mathfrak{p}\text{-rank}(x = x) \leq \omega$.

Now assume for a contradiction that $\varphi(x, \vec{b})$ is not a small set. By 2.16 there is some open interval $\mathbb{I}_{\vec{b}}$ such that $\mathbb{D}_{\vec{b}}$ is cosmall in $\mathbb{I}_{\vec{b}}$, that is, $\mathbb{D}_{\vec{b}} \cap \mathbb{I}_{\vec{b}} = \mathbb{I}_{\vec{b}} \setminus \mathbb{S}_{\vec{b}}$ where $\mathbb{S}_{\vec{b}}$ is a small set. Suppose that $\theta(\vec{y}, \vec{c})$ is such that for any $\vec{b}_1, \dots, \vec{b}_k$, each realizing $\theta(\vec{y}, \vec{c})$, one has

$$\mathbb{D}_{\vec{b}_1} \cap \dots \cap \mathbb{D}_{\vec{b}_k} = \emptyset.$$

Thus we have

$$\emptyset = \bigcap_{1 \leq i \leq k} (\mathbb{D}_{\vec{b}_i} \cap \mathbb{I}_{\vec{b}_i}) = \bigcap_{1 \leq i \leq k} \mathbb{I}_{\vec{b}_i} \setminus \bigcup_{1 \leq i \leq k} \mathbb{S}_{\vec{b}_i}$$

Then it is not hard to see that

$$\mathbb{J} := \mathbb{I}_{\vec{b}_1} \cap \dots \cap \mathbb{I}_{\vec{b}_k} = \emptyset.$$

For if this were not the case, \mathbb{J} would be an open interval contained in the small set $\mathbb{S}_{\vec{b}_1} \cup \dots \cup \mathbb{S}_{\vec{b}_k}$, which is impossible, by Corollary 2.19.

Thus, if $\psi(x, \vec{b})$ defines $\mathbb{I}_{\vec{b}}$, we see that $\psi(x, \vec{b})$ also \mathfrak{p} -divides. But since intervals are \mathcal{L} -definable, this contradicts the previous lemma. Thus we conclude that $\mathfrak{p}\text{-rank}(x = x)$ is no greater than ω .

It remains to show that if R has a field structure, then $\mathfrak{p}\text{-rank}(x = x)$ is precisely ω . Note that as G is small, R is an infinite dimensional $\text{dcl}(G)$ -vector space. Choose $(c_i)_{i \in \mathbb{N}}$ independent vectors. Considering

$$c_1 G + \dots + c_{n-1} G + c_n g,$$

and noting that one gets 2-inconsistency as one varies g through G , it is clear that

$$\mathbb{V}_{\vec{c}}^n := c_1 G + \dots + c_{n-1} G + c_n G$$

has \mathfrak{b} -rank n . As each \mathbb{V}_c^n is a subset of R , $\mathfrak{b}\text{-rank}(R) \geq \omega$. \square

Note that we have not only shown that \mathfrak{R} is super-rosy, but the following:

Corollary 3.5. *Any formula $\varphi(x, \vec{b})$ that \mathfrak{b} -divides defines a small subset of R .*

This will allow us to show that, in certain cases, small closure gives rise to a pregeometry in Section 7.

Finally, we should point out the following two corollaries:

Corollary 3.6. *Dense pairs of o -minimal structures (with at least a group structure) are superrosy. If the o -minimal structure is an expansion of a real closed field, the \mathfrak{b} -rank of the pair is ω .*

Proof. See [2] to see a proof that dense pairs satisfy the hypotheses of Theorem 1.3. \square

For the next corollary, we need a definition and a fact from [7]:

Definition 3.7. An expansion of $(\mathbb{R}, <)$ is said to have *o -minimal open core* if the reduct generated by the definable open sets is o -minimal.

Fact 3.8. *An expansion of $(\mathbb{R}, +, \cdot)$ has o -minimal open core if and only if each definable open subset of \mathbb{R} has finitely many connected components.*

Corollary 3.9. *An expansion of $(\mathbb{R}, +, \cdot)$ which satisfies the hypotheses of Theorem 1.3 has o -minimal open core.*

Proof. For a contradiction, let \mathbb{D} be definable, open, and with infinitely many connected components. We may assume that $\mathbb{D} \subset (a, \infty)$. We note that that given $d \in \mathbb{D}$, the connected component of \mathbb{D} containing d is definable, say by $\varphi(x, d)$. Being in the same connected component is a definable equivalence relation, call it E . Thus the connected component of d may just as easily be defined by $\tilde{\varphi}(x, d/E)$. As d/E varies through the sort \mathbb{D}/E , $\tilde{\varphi}(x, d/E)$ \mathfrak{b} -divides. But $\tilde{\varphi}(x, d/E)$ is an interval, and hence \mathcal{L} -definable. This contradicts Lemma 3.4. \square

4. IMAGINARIES

Pillay, building on ideas of Lascar, showed that a strongly minimal theory where the algebraic closure of the empty set is infinite eliminates imaginaries down to finite sets (see e.g. [6]). What follows is the same argument, with small replacing finite, and it shows that \mathfrak{R} eliminates imaginaries down to small sets.

In this section, we assume that (R, G) satisfies all the hypotheses of Theorem 1.4. That is, we add to the assumptions of the last section, the assumption that given any set A , and I any interval defined over A , that $\text{scl}(A) \cap I$ is not contained in any small set.

Proposition 4.1. *Let $\varphi(\vec{x}, \vec{y})$ define an equivalence relation, E , and let e be an element of the sort $R_{\vec{x}}/E$. Then there is an element, \vec{d} , of $R_{\vec{x}}$ such that $e = \vec{d}/E$ and $\vec{d} \in \text{scl}(e)$.*

Proof. Let $\pi : R^n \rightarrow R^n/E$ be the quotient map, and consider \mathbb{D}_1 defined by

$$\exists x_2, \dots, x_n \pi(x_1, x_2, \dots, x_n) = e.$$

In the case that \mathbb{D}_1 is small, any element of \mathbb{D}_1 is in $\text{scl}(e)$; let d_1 be any such element. Otherwise, there is some interval such that \mathbb{D}_1 is cosmall in that interval.

By our assumption on the small closure, it is not possible that $\text{scl}(e)$ is contained in \mathbb{D}_1^c . Let d_1 be some element of $\text{scl}(e) \cap \mathbb{D}_1$.

Proceed inductively and define \mathbb{D}_i as

$$\exists x_{i+1}, \dots, x_n \pi(d_1, \dots, d_{i-1}, x_i, x_{i+1}, \dots, x_n) = e$$

and consider the cases of \mathbb{D}_i small, or not, as above, to get $\vec{d} := (d_1, \dots, d_n)$. Then $d_i \in \text{scl}(\vec{e}, d_1, \dots, d_{i-1})$. By choice of d_1, \dots, d_{i-1} , together with the fact that $\text{scl} : \mathcal{P}(R) \rightarrow \mathcal{P}(R)$ is a closure operator, this implies that $d_i \in \text{scl}(e)$. □

Now we may prove our elimination of imaginaries result:

Theorem 1.4. *Enlarge \mathfrak{X} to $\tilde{\mathfrak{X}}$ by adding sufficiently many sorts of \mathfrak{X}^{eq} so that $\tilde{\mathfrak{X}}$ has a code for every basic small subset of R^k . Then $\tilde{\mathfrak{X}}$ eliminates imaginaries.*

Proof. Take $e \in R^{\text{eq}}$. We want to find $c \in \tilde{R}$ such that c is interdefinable with e . Take \vec{d} such that $\pi(\vec{d}) = e$ and $\vec{d} \in \text{scl}(e)$. Thus \vec{d} is in a basic small set, \mathbb{D} , defined over e ; let c be the code for $\mathbb{D} \cap \pi^{-1}(e)$. Clearly, c is defined over e . But e is defined over any element of $\mathbb{D} \cap \pi^{-1}(e)$, and thus over c as well. □

5. GROUPS WITH THE MANN PROPERTY

We start by defining the Mann property for multiplicative subgroups of fields. Let K be a field, and G a subgroup of K^\times . For $a_1, \dots, a_n \in K$, a solution (g_1, \dots, g_n) of $a_1x_1 + \dots + a_nx_n = 1$ in G is said to be *nondegenerate* if $\sum_{i \in I} a_i g_i \neq 0$ for every non-empty subset I of $\{1, \dots, n\}$. We say G has the *Mann property* if for every a_1, \dots, a_n from K , the equation $a_1x_1 + \dots + a_nx_n = 1$ has finitely many nondegenerate solutions in G .

Prior to this section, we have assumed that (R, G) was as in Theorem 1.3. In this section we instead prove that (R, G) as in Theorem 1.1 satisfy the hypotheses of Theorem 1.3. That is, we assume that R is a real closed field and G is a dense subgroup of $R^{>0}$ with the Mann property and such that for each p , the p th powers in G have finite index in G .

As noted in the introduction, most of the results about groups with the Mann property that we need are found in [3]. For instance, we have the following:

Fact 5.1. *By of Lemma 6.1 of [3], if (R, G) satisfies the conditions of Theorem 1.1, then G is small.*

Fact 5.2. *By Theorem 7.5 of [3], if (R, G) satisfies the conditions of Theorem 1.1, then any definable subset of \mathfrak{X} is a boolean combination of basic sets.*

However, we will need to strengthen the quantifier elimination results obtained there.

In the rest of this section q is of the form p^m , where p is a prime number and $m \in \mathbb{N}$.

For each q and $\vec{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n$ let $D_{q, \vec{k}}(\vec{x})$ be the formula

$$\mathcal{G}(x_1) \wedge \dots \wedge \mathcal{G}(x_n) \wedge \exists y (\mathcal{G}(y) \wedge x_1^{k_1} \dots x_n^{k_n} = y^q).$$

Note that $D_{q, (0, \dots, 0)}(R_{\vec{x}})$ is all of G^n , and for any $g \in G$, there is $\vec{h} \in G^n$ such that $D_{q, \vec{1}\vec{k}}(g, R_{\vec{x}})$ equals $\vec{h}D_{q, \vec{k}}(R_{\vec{x}})$.

We will write $G^{[n]}$ to denote the elements of G that have n th roots in G .

Proposition 5.3. *Let $\mathbb{D} \subseteq G^n$ be definable in (R, G) , then \mathbb{D} is a boolean combination of sets of the form $\mathbb{F} \cap \vec{g}D_{q, \vec{k}}(R_{\vec{x}})$, where \mathbb{F} is a semialgebraic set, $\vec{g} \in G^n$, q is as above, and $\vec{k} \in \mathbb{Z}^n$.*

Before proving this proposition, we recall some results from [3] that are used in the proof of it.

Let (R_1, G_1) and (R_2, G_2) be two $|R|^+$ -saturated elementary extensions of (R, G) . Then in the proof of Theorem 7.1 of [3], the authors construct a back and forth system \mathcal{I} , between (R_1, G_1) and (R_2, G_2) , consisting of isomorphisms $\iota : (R'_1, G'_1) \rightarrow (R'_2, G'_2)$ where R'_i is a real closed ordered subfield of R_i of cardinality $< |R|$, $G'_i \subseteq R_i^{>0}$ is a pure subgroup of G_i containing G , and R'_i and $\mathbb{Q}(G_i)$ are algebraically free over $\mathbb{Q}(G'_i)$ for $i = 1, 2$.

We also need the following lemma from [3].

Lemma 5.4. *Let R be a real closed field with a subfield E and let $H \subseteq R^{>0}$ be a subgroup satisfying the Mann property. Suppose that H' is a subgroup of H such that for all $a_1, \dots, a_n \in E^\times$ the equation $a_1x_1 + \dots + a_nx_n = 1$ has the same nondegenerate solutions in H' as in H . Then for any $h \in H$, if h is algebraic over $E(H')$ of degree d , then $h^d \in H'$.*

Now we prove Proposition 5.3.

Proof. By standard model theoretic arguments (see for instance 8.4.1 of [5]), it is enough to prove the following:

Claim. Let (R_1, G_1) and (R_2, G_2) be two $|R|^+$ -saturated elementary extensions of (R, G) . Take $\vec{g}_1 \in G_1^n$ and $\vec{g}_2 \in G_2^n$ such that for any formula $\varphi(\vec{x})$ in the language of ordered rings with parameters in R , for any $g \in G$, and for any q, \vec{k} as above, we have

$$(R_1, G_1) \models \varphi(\vec{g}_1) \wedge D_{q, 1\vec{k}}(g, \vec{g}_1) \text{ iff } (R_2, G_2) \models \varphi(\vec{g}_2) \wedge D_{q, 1\vec{k}}(g, \vec{g}_2).$$

Then $(R_1, G_1, \vec{g}_1) \equiv_R (R_2, G_2, \vec{g}_2)$.

Proof of the claim. By the remarks made before the proof, there is a back and forth system \mathcal{I} between (R_1, G_1) and (R_2, G_2) . It suffices to prove that there is an element ι of \mathcal{I} taking \vec{g}_1 to \vec{g}_2 .

Since \vec{g}_1 and \vec{g}_2 satisfy the same ordered field type over R , there is a ordered field isomorphism $\iota : R'_1 \rightarrow R'_2$, mapping \vec{g}_1 to \vec{g}_2 equal to the identity on R , where R'_i is the real closure of $R(\vec{g}_i)$ for $i = 1, 2$.

Consider $G'_i := R'_i \cap G_i$. We wish to show that $G'_i = G(\vec{g}_i) := \{(g\vec{g}_i^{\vec{k}})^{1/m} : g \in G, \vec{k} \in \mathbb{Z}^n, m \in \mathbb{N}, g\vec{g}_i^{\vec{k}} \in G_i^{[m]}\}$. It is clear that $G'_i \supseteq G(\vec{g}_i)$.

We use Lemma 5.4 to show $G'_i \subseteq G(\vec{g}_i)$. To do this we need to check that for all $a_1, \dots, a_n \in R$, if $a_1x_1 + \dots + a_nx_n = 1$ has a nondegenerate solution in G_i , then this solution lies in $G(\vec{g}_i)$. But since $(R, G) \preceq (R_i, G_i)$, such a solution lies even in G . Now applying Lemma 5.4, we see that if $g \in G_i$ is algebraic of degree d over $R(G(\vec{g}_i))$, then g^d is in $G(\vec{g}_i)$ and thus g itself is in $G(\vec{g}_i)$.

Now we wish to show that $\iota(G'_1) = G'_2$. An element of G'_1 is of the form $(g\vec{g}_1^{\vec{k}})^{1/m}$ for some $g \in G, \vec{k} \in \mathbb{Z}^n, m \in \mathbb{N}$. Note $\iota((g\vec{g}_1^{\vec{k}})^{1/m}) = (g\vec{g}_2^{\vec{k}})^{1/m}$, and by our

assumption on \vec{g}_i , $(g\vec{g}_1^{\vec{k}})$ is in $G_1^{[m]}$ if and only if $(g\vec{g}_2^{\vec{k}})$ is in $G_2^{[m]}$. Thus ι is an isomorphism between (R'_1, G'_1) and (R'_2, G'_2) .

It remains to show that R'_i and $\mathbb{Q}(G'_i)$ are algebraically free over $\mathbb{Q}(G'_i)$ and G'_i is a pure subgroup of G_i . The first follows from the assumption that (R_i, G_i) is an elementary extension of (R, G) , and G'_i is a pure subgroup of G_i , since it equals $G\langle\vec{g}_i\rangle$. \square

Remark 5.5. *Note that the proof of Proposition 5.3 does not require that the subgroup of p th powers has finite index. With this assumption, we see that in addition, the subgroup of q th powers is of finite index in G and therefore $D_{q, \vec{k}}(R_{\vec{x}})$ is of finite index in G^n . So $G^n \setminus D_{q, \vec{k}}(R_{\vec{x}})$ is a finite union of cosets of $D_{q, \vec{k}}(R_{\vec{x}})$.*

We also have the following lemma.

Lemma 5.6. *For any q , and $\vec{k} \in \mathbb{Z}^n$, $D_{q, \vec{k}}(R_{\vec{x}})$ is dense in G^n .*

Proof. We show that for any q , and $\vec{k} \in \mathbb{Z}^n$, $D_{q, \vec{k}}(R_{\vec{x}}) \supseteq (G^{[q]})^n$, which is enough to prove the lemma, as $(G^{[q]})^n$ is dense in G^n . So let $(g_1^q, \dots, g_n^q) \in (G^{[q]})^n$. Then

$$(g_1^q)^{k_1} \dots (g_n^q)^{k_n} = (g_1^{k_1})^q \dots (g_n^{k_n})^q = (g_1^{k_1} \dots g_n^{k_n})^q \in G^{[q]}.$$

Thus $(g_1^q, \dots, g_n^q) \in D_{q, \vec{k}}(R_{\vec{x}})$. \square

Corollary 5.7. *Each $D_{q, \vec{k}}(R_{\vec{x}})$ is a finite union of cosets of $(G^{[q]})^n$. Moreover, for any $\mathbb{D} \subset G^n$ there is $d \in \mathbb{N}$ such that \mathbb{D} is a finite union of sets of the form $\mathbb{F} \cap \vec{g}(G^{[d]})^n$ where \mathbb{F} is semialgebraic.²*

Proof. By the proof of Lemma 5.6, we have that $(G^{[q]})^n$ is a subgroup of $D_{q, \vec{k}}(R_{\vec{x}})$. Since $(G^{[q]})^n$ is finite index in G^n , it is also finite index in $D_{q, \vec{k}}(R_{\vec{x}})$.

Next note that if d is the least common multiple of d_1, d_2 , then $G^{[d_1]} \cap G^{[d_2]} = G^{[d]}$. Thus, given any finite number of cosets of $(G^{[d_i]})^n$ for various d_i , one may replace them by a finite number of cosets of $(G^{[d]})^n$, where d is the least common multiple of the d_i . Using this observation, the reader may easily check that for each $\mathbb{D} \subset G^n$ there is $d \in \mathbb{N}$ such that \mathbb{D} is a finite union of sets of the form $\mathbb{F} \cap \vec{g}(G^{[d]})^n$ where \mathbb{F} is semialgebraic. \square

Now we are in a position to prove the first of our main results.

Theorem 1.1. *$\mathfrak{R} = (R, G)$ is super-rosy of \mathfrak{p} -rank equal to ω and \mathfrak{p} -rank of G is 1.*

Proof. Since super-rosiness and \mathfrak{p} -rank are properties of the theory, we may assume that (R, G) is sufficiently saturated. Conditions (1) and (2) of Theorem 1.3 are clear; we will show (3) for (R, G) in a language expanded by naming each element of some model. Consider $\mathbb{D} \subseteq G^n$. First, we wish to show that $\mathbb{D} = \mathbb{E} \cap \mathbb{S}$, where \mathbb{E} is semialgebraic and \mathbb{S} is a dense subset of G^n . For the purposes of this proof, we refer to such sets as *nice*.

We have established, in the previous corollary, that $\mathbb{D} = \bigcup_{i=1}^m \mathbb{E}_i \cap \mathbb{S}_i$, where each \mathbb{E}_i is semialgebraic, and each \mathbb{S}_i is of the form $\vec{g}(G^{[d]})^n$, and, in particular, each \mathbb{S}_i is dense in G^n . Thus \mathbb{D} is a finite union of nice sets. We wish to show that a finite union of nice sets is nice. Consider $(\mathbb{E}_1 \cap \mathbb{S}_1) \cup (\mathbb{E}_2 \cap \mathbb{S}_2)$. Let $\widetilde{\mathbb{E}}_1 := \mathbb{E}_1 \setminus \mathbb{E}_2$ and

²The authors thank Lou van den Dries for pointing out this Corollary.

$\tilde{\mathbb{E}}_2 := \mathbb{E}_2 \setminus \mathbb{E}_1$. Let $\tilde{\mathbb{S}}_1 := \mathbb{S}_1 \setminus \mathbb{S}_2$ and $\tilde{\mathbb{S}}_2 := \mathbb{S}_2 \setminus \mathbb{S}_1$. Let $\mathbb{E} := (\mathbb{E}_1 \cup \mathbb{E}_2)$ and let $\mathbb{S} = (\mathbb{S}_1 \cup \mathbb{S}_2) \setminus ((\tilde{\mathbb{E}}_2 \cap \tilde{\mathbb{S}}_1) \cup (\tilde{\mathbb{E}}_1 \cap \tilde{\mathbb{S}}_2))$. Note that

$$(\mathbb{E}_1 \cap \mathbb{S}_1) \cup (\mathbb{E}_2 \cap \mathbb{S}_2) = \mathbb{E} \cap \mathbb{S}.$$

Thus we want to show that \mathbb{S} is dense in G^n .

Suppose that \mathbb{S} is not dense in G^n . Then there is an semialgebraic open $\mathbb{U} \subseteq (R^{>0})^n$ such that $\mathbb{S} \cap \mathbb{U} = \emptyset$. Thus $\mathbb{S}_1 \cap \mathbb{U} \subseteq \tilde{\mathbb{E}}_2 \cap \tilde{\mathbb{S}}_1$ and $\mathbb{S}_2 \cap \mathbb{U} \subseteq \tilde{\mathbb{E}}_1 \cap \tilde{\mathbb{S}}_2$. Since \mathbb{S}_2 is dense, the closure of $\mathbb{S}_2 \cap \mathbb{U}$ equals the closure of \mathbb{U} , and is contained in the closure of $\tilde{\mathbb{E}}_1$. Thus, $\tilde{\mathbb{E}}_1$ must contain all of \mathbb{U} except for a semialgebraic set, \mathbb{D}_1 , of dimension less than n . Likewise there is \mathbb{D}_2 such that $\mathbb{U} \setminus \mathbb{D}_2 \subseteq \tilde{\mathbb{E}}_2$. Thus $\mathbb{U} \setminus (\mathbb{D}_1 \cup \mathbb{D}_2)$ is contained in $\tilde{\mathbb{E}}_1 \cap \tilde{\mathbb{E}}_2 = \emptyset$, a contradiction.

Finally we note that by Corollary 5.7, if $\mathbb{D} \subseteq G^n$, then $\mathbb{D} = \bigcup_{i < k} (\mathbb{E}_i \cap \mathbb{S}_i)$ with each \mathbb{E}_i a semialgebraic set and each \mathbb{S}_i a coset of $(G^{[d]})^n$. Since $(G^{[d]})^n$ is a subgroup of finite index, any model has representatives of each coset, and thus, after naming the elements any model, each \mathbb{S}_i becomes \emptyset -definable, and we may apply Theorem 1.3 to get that (R, G) in this expanded language is super-rosy of \mathfrak{p} -rank equal to ω and \mathfrak{p} -rank of G is 1. Since \mathfrak{p} -rank is invariant under expansions of the language by constants, we are done. \square

In [7], the question is raised whether $(\mathbb{R}, 2^{\mathbb{Z}}3^{\mathbb{Z}})$ has o-minimal open core. We are now in a position to give an affirmative answer to this question.

Corollary 5.8. *If (\mathbb{R}, G) is an expansion of the real field by a predicate for a dense multiplicative subgroup of $\mathbb{R}^{>0}$ with the Mann property, then (\mathbb{R}, G) has o-minimal open core.*

Proof. By Corollary 3.9. \square

To prove the second main result, that adding codes for the small sets definable in \mathfrak{R} is sufficient to eliminate imaginaries, we must verify our assumptions at the beginning of Section 4: that given any set of parameters A , and any interval I defined over A , the small closure of A intersect I is not contained in any small set. To do this, we must first perform some \mathfrak{p} -rank calculations within \mathfrak{R} .

Definition 5.9. For $n > 0$ we define G^{+n} inductively as

$$G^{+1} := G \cup \{0\},$$

$$\text{and } G^{+(n+1)} := (G \cup \{0\}) + G^{+n}.$$

Proposition 5.10. *The \mathfrak{p} -rank of G^{+n} is n .*

Proof. Consider the map $f : G^n \rightarrow G^{+n}$ given by $f(\vec{g}) = g_1 + \cdots + g_n$. We have \mathfrak{p} -rank of G^{+n} is less than or equal to n , since f is surjective.

For the converse, define $G_I^n := \{\vec{g} \in G^n : \sum_{i \in I} g_i = 0\}$ for any nonempty subset I of $\{1, \dots, n\}$. Note that G_I^n is the image of G^{n-1} under a definable map, thus is of \mathfrak{p} -rank at most $n-1$. Now define

$$G_{\text{nd}}^n := G^n \setminus \bigcup_{\emptyset \neq I \subseteq \{1, \dots, n\}} G_I^n.$$

Note that \mathfrak{p} -rank of G_{nd}^n is n , and by the Mann property, the restriction of f to G_{nd}^n has finite fibers. Therefore, by 3.1, \mathfrak{p} -rank of G^{+n} is n . \square

Proposition 5.11. *Let A be any set, and I any interval defined over A . Then $\text{scl}(A) \cap I$ is not contained in any small set.*

Proof. Note that $\text{scl}(A)$ contains $\text{scl}(\emptyset)$ which in turn contains G^{+n} . First we show that

$$\bigcup_{n>0} G^{+n}$$

is not contained in any small set. Assume it is contained in a small set \mathbb{S} . Since \mathbb{S} is G -bound, there is a map $f : R^k \rightarrow R$ such that $\mathbb{S} \subseteq f(G^k)$. Therefore by Propositions 3.1 and 3.3, we have \mathfrak{b} -rank of \mathbb{S} is at most k , and thus, for each n , G^{+n} has \mathfrak{b} -rank at most k contradicting Proposition 5.10.

Let $I = (b, c)$. Now let $f : R \rightarrow (b, c)$ be a definable bijection. Note that $f(\bigcup_{n>0} G^{+n})$ is contained in $\text{scl}(A) \cap I$. If $f(\bigcup_{n>0} G^{+n})$ were contained in some small set, say \mathbb{S} , then $f^{-1}(\mathbb{S})$ would be a small set containing $\bigcup_{n>0} G^{+n}$, a contradiction. \square

Now we have proven the second of main results:

Theorem 1.2 *If one enlarges (R, G) by adding sufficiently many sorts of $(R, G)^{\text{eq}}$ so that the resulting structure has a code for every basic small subset of R^k , then this structure eliminates imaginaries.*

6. THE STRUCTURE $R^{>0}/G$

In this section we assume that R has a field structure.

Proposition 6.1. *Let $C \subset R$ and let $a, b \in R$ be such that $a, b \notin \text{scl}(C)$. Then for every formula $\varphi(x, \vec{c})$ in $\text{tp}(a/C)$ there is $b' \in R$ such that $b'/G = b/G$ and $b' \in \varphi(R, \vec{c})$.*

Proof. We may assume that $C = \text{dcl}(C)$. Let $\varphi(x, \vec{c}) \in \text{tp}(a/C)$. By Lemma 2.16 there is a partition $\{c_0, \dots, c_n\}$ of R , where $c_i \in C$ for $i \leq n$ such that $\varphi(x, \vec{c})$ is small or cosmall when restricted to (c_i, c_{i+1}) . Say $a \in (c_i, c_{i+1})$. Since $a \notin \text{scl}(C)$, $\varphi(R, \vec{c})$ is cosmall in (c_i, c_{i+1}) . Since $b \neq 0$, there is $t \in R$ such that $tb = a$. Furthermore, since multiplication by b is a continuous function, and since G is dense in R , we can find $g \in G$ such that $b' = gb \in (c_i, c_{i+1})$. We may choose g \mathfrak{b} -independent from b over C . Since $b \notin \text{scl}(C \cup \{g\})$ and multiplication by g is a definable bijection of R , we have that $b' \notin \text{scl}(C \cup \{g\})$ and thus $\varphi(x, \vec{c}) \in \text{tp}(b'/C)$. \square

Corollary 6.2. *Let $a, b \in R$ be such that $a, b \notin \text{scl}(A)$. Let $a_G = a/G$, $b_G = b/G$. Then for any set A such that a_G and b_G are \mathfrak{b} -independent from A , $\text{tp}(a_G/A) = \text{tp}(b_G/A)$.*

Proof. We may assume that a and b are independent from A . By the previous proposition for every formula $\varphi(x, \vec{c})$ in $\text{tp}(a/A)$ we can find $b' \in R$ such that $b'/G = b_G$ and $b' \in \varphi(R, \vec{c})$. This implies that $\text{tp}(a_G/A) = \text{tp}(b_G/A)$. \square

Given any subset $C \subset \mathfrak{R}$, there is a unique type in $R^{>0}/G$ over C that contains only large sets. Thus the group $R^{>0}/G$ is definably connected (in the sense of having no proper definable subgroups of finite index) and all definable subsets of $R^{>0}/G$ are small or cosmall.

Assume now that R is uncountable and G is countable. Then the definable small sets are countable. This raises the following question:

Question 6.3. *Is $R^{>0}/G$ quasi-minimal?*

In [11], Zilber defines a quasi-minimal excellent class, as a class of structures closed under isomorphism, where each definable set is countable or co-countable, and with a closure operator satisfying three assumptions. When, in addition, the closure operator satisfies the exchange property, he obtains that the class is categorical in every uncountable cardinal. We have that each definable set is countable or co-countable, and small closure satisfies exchange and can easily be seen to satisfy the first of Zilber's three assumptions. However, we have been unable to verify that the other two assumptions hold.

Even without the assumption that G is countable, we may ask the following, less ambitious, question:

Question 6.4. *Is $R^{>0}/G$ superstable?*

There is no obvious order definable within $R^{>0}/G$, and if $R^{>0}/G$ does not have the order property, it must be superstable, as \mathfrak{b} -forking agrees with forking in stable theories.

7. THE $U^{\mathfrak{b}}$ -RANK

Throughout this section, \mathfrak{A} denotes a structure satisfying the hypotheses of Theorem 1.3.

In [1] Buechler used infinite dimensional pairs to study the geometric properties of a strongly minimal sets. He showed the pair has Morley rank one iff the strongly minimal set is trivial, Morley rank two iff the strongly minimal set is locally modular non trivial and ω otherwise. These results were generalized by Vassiliev in [10] to the setting of simple theories using lovely pairs to analyze SU rank one pregeometries. Dense pairs of o-minimal structures were studied by van den Dries in [2], where he showed they satisfy the hypotheses of Theorem 1.3. In what follows below, we show that the same relationship exists between the pregeometry of a o-minimal structure, and that of the corresponding dense pair (though, of course, here the information yielded by the dense pair is already known).

Peterzil and Starchenko [9] showed that locally every o-minimal structure behaves as an expansion of a field, an ordered vector space, or is trivial. In the analysis that follows below, we will deal with two cases: when \mathfrak{A} includes a field structure and when $\mathfrak{A}|_{\mathcal{L}}$ is an ordered abelian group with no additional structure.

Recall that the $U^{\mathfrak{b}}$ -rank "counts" the number of times the type can \mathfrak{b} -fork and that 1-types in o-minimal structures have $U^{\mathfrak{b}}$ -rank at most one.

Lemma 7.1. *Let $g \in G$ and let $C \subset R$. Then $U^{\mathfrak{b}}(\text{tp}(g/C)) \leq 1$ and equality holds iff $g \notin \text{dcl}(C)$.*

Proof. It follows from Theorem 1.3. □

7.1. Field case. Now assume that $\mathfrak{A}|_{\mathcal{L}}$ has a definable field structure. Then, as G is small, R is an infinite dimensional $\text{dcl}(G)$ -vector space and we fix a countable family $(c_i)_{i \in \omega}$ of linearly independent vectors.

Definition 7.2. Let $g_1, \dots, g_n \in G$ and let $A \subset R$. We say that $\{g_1, \dots, g_n\}$ is an A -independent set if $U^{\mathfrak{b}}(\text{tp}(g_1, \dots, g_n/A)) = n$.

Lemma 7.3. *Let $g_1, \dots, g_n \in G$ and let $C = \{c_1, \dots, c_n\}$. Then*

$$U^{\mathfrak{b}}(\text{tp}(c_1g_1 + \dots + c_ng_n/C)) \leq n$$

and equality holds iff $\{g_1, \dots, g_n\}$ is a C -independent set.

Proof. Clearly $c_1g_1 + \dots + c_ng_n \in \text{dcl}(\{g_1, \dots, g_n, c_1, \dots, c_n\})$, so by additivity of the rank and the previous lemma,

$$U^{\mathfrak{b}}(\text{tp}(c_1g_1 + \dots + c_ng_n/C)) \leq U^{\mathfrak{b}}(\text{tp}(g_1, \dots, g_n/C)) \leq n.$$

Furthermore since $C = \{c_1, \dots, c_n\}$ is a set of linearly independent vectors, there is only one solution in G^n for the equation $c_1x_1 + \dots + c_nx_n = c_1g_1 + \dots + c_ng_n$, so $g_1, \dots, g_n \in \text{dcl}(g_1c_1 + \dots + g_nc_n, C)$. If $\{g_1, \dots, g_n\}$ is a C -independent set, we get $U^{\mathfrak{b}}(\text{tp}(c_1g_1 + \dots + c_ng_n/C)) = n$. \square

Proposition 7.4. *Let $a \notin \text{scl}(\emptyset)$, then $U^{\mathfrak{b}}(\text{tp}(a)) = \omega$.*

Proof. By Theorem 1.3 (and Fact 1.12), $U^{\mathfrak{b}}(\text{tp}(a)) \leq \omega$.

Now we will show that $\text{tp}(a/\emptyset)$ has forking extensions of $U^{\mathfrak{b}}$ -rank n for every n . Let $C = \{c_1, \dots, c_n\}$ and without loss of generality assume that C is \mathfrak{b} -independent from a . Let $g_1, \dots, g_n \in G$ and assume that $\{g_1, \dots, g_n\}$ is a $C \cup \{a\}$ -independent set. Let $b = a + c_1g_1 + \dots + c_ng_n$. Then $a, b \notin \text{scl}(\{c_1, \dots, c_n\})$. Thus $U^{\mathfrak{b}}(\text{tp}(c_1g_1 + \dots + c_ng_n/C \cup \{b\})) = n$ and since a and $c_1g_1 + \dots + c_ng_n$ are interdefinable over b , $U^{\mathfrak{b}}(\text{tp}(a/C \cup \{b\})) = n$. Thus $U^{\mathfrak{b}}(\text{tp}(a)) = \omega$. \square

Corollary 7.5. *If $\mathfrak{R}|_{\mathcal{L}}$ has a definable field structure and $a \in \text{scl}(B) \setminus \text{scl}(C)$, then $a \not\downarrow_C^{\mathfrak{b}} B$.*

Proof. We may assume $C = \emptyset$, as our hypotheses remain true after adding parameters to the language. Since $a \in \text{scl}(B)$, some formula in $\text{tp}(a/B)$ defines a G -bound set, and Lemma 7.3 implies that $U^{\mathfrak{b}}(a/B)$ is finite. On the other hand, $U^{\mathfrak{b}}(a) = \omega$ by Lemma 7.4. \square

7.2. Pairs of groups with no additional structure. Assume now that $\mathcal{L} = \{+, 0, <\}$. Thus $\mathfrak{R}|_{\mathcal{L}}$ is a divisible ordered abelian group. Furthermore suppose that G a subgroup of R .

Definition 7.6. Let $n > 0$ and let $G/n = \{r \in R : nr \in G\}$.

Lemma 7.7. *The group G/n has \mathfrak{b} -rank one.*

Proof. Recall that G has \mathfrak{b} -rank one. As R is divisible and torsion-free, multiplication by n is a definable bijection between G/n and G , and thus the \mathfrak{b} -rank of G/n is one. \square

Proposition 7.8. *$a \in \text{scl}(B)$ if and only if there is $b \in \text{dcl}(B)$ and $n \in \mathbb{N}^{>0}$ such that $a \in b + G/n$.*

Proof. Right to left is clear.

Now assume that $a \in \text{scl}(B)$. By Proposition 2.16, a is contained in \mathbb{S} , a basic small set defined over B . Let $\exists \vec{y}(\mathcal{G}(\vec{y}) \wedge \varphi(x, \vec{y}))$ be a formula defining \mathbb{S} . For each \vec{g} , $\varphi(R, \vec{g})$ is a finite union of points and intervals. However, if for any \vec{g} in G^k ,

$\varphi(R, \vec{g})$ contains a non-empty open interval, then \mathbb{S} is not small. Thus, we may reduce to the case where $\varphi(x, \vec{y})$ is $x = f(\vec{y})$, where

$$f(\vec{y}) = b + \sum_{i=1}^k \frac{m_i}{n_i} y_i$$

for some $b \in \text{dcl}(B)$, $m_i \in \mathbb{Z}$ and $n_i \in \mathbb{N}$. Let n be the least common multiple of the n_i . Thus $f(G^k)$ is contained in $b + G/n$, and $a \in f(G^k)$. \square

Proposition 7.9. *Let $a \in R$ be such that $a \notin \text{scl}(\emptyset)$. Then $U^{\mathfrak{b}}(\text{tp}(a)) = 2$.*

Proof. By Proposition 7.8, every small subset of R has \mathfrak{b} -rank at most one, and by Corollary 3.5, a \mathfrak{b} -forking extension of $\text{tp}(a)$ must include a formula defining a small set. Thus $U^{\mathfrak{b}}(\text{tp}(a)) \leq 2$. It is easy to see that for $g \in G$, with $\text{tp}(g)$ non-algebraic, and $g \perp^{\mathfrak{b}} a$, we get $U^{\mathfrak{b}}(\text{tp}(a)) = U^{\mathfrak{b}}(\text{tp}(a/g)) = U^{\mathfrak{b}}(\text{tp}(a + g/g))$. Now we claim that $a + g \perp^{\mathfrak{b}} g$. If not, by Corollary 3.5 we would have $a + g \in \text{scl}(g) = \text{scl}(\emptyset)$, and thus $a + g \in c + G/n$ for some $c \in \text{dcl}(\emptyset)$, by Proposition 7.8. But then $a + g$, and hence a , would be in $\text{scl}(\emptyset)$, a contradiction. Thus $U^{\mathfrak{b}}(a) = U^{\mathfrak{b}}(a + g/g) = U^{\mathfrak{b}}(a + g)$, and it suffices to show that $U^{\mathfrak{b}}(a + g) = 2$.

Consider the chain $\text{tp}(a + g/\emptyset) \subset \text{tp}(a + g/a) \subset \text{tp}(a + g/a, g)$. If we show that this is a \mathfrak{b} -forking chain we will have shown that $U^{\mathfrak{b}}(a + g) \geq 2$, and thus equal to 2. First note that $\text{tp}(a + g/a)$ contains a formula saying $x \in G + a$. This formula is true of $a + g$ and \mathfrak{b} -divides over the empty set. Thus, $\text{tp}(a + g/a)$ is a \mathfrak{b} -forking extension of $\text{tp}(a + g)$.

Second, note that $\text{tp}(a + g/a, g)$ is algebraic, and hence to show that it is a \mathfrak{b} -forking extension of $\text{tp}(a + g/a)$, it suffices to show that the latter type is not algebraic. But we chose $g \perp^{\mathfrak{b}} a$. Thus $\text{tp}(g/a)$ is not algebraic, and neither is $\text{tp}(a + g/a)$. \square

Now we get a corollary analogous to Corollary 7.5:

Corollary 7.10. *If $\mathfrak{R}|_{\mathcal{L}}$ is an ordered group with no additional structure, and $a \in \text{scl}(B) \setminus \text{scl}(C)$, then $a \not\perp_C^{\mathfrak{b}} B$.*

Proof. By the previous proposition (after adding C to the language), we see that $U^{\mathfrak{b}}(a/C) = 2$. On the other hand, by Proposition 7.8, we see that a belongs to a set of \mathfrak{b} -rank one defined over B , namely a coset of G/n for some n . Thus $U^{\mathfrak{b}}(a/B)$ is either zero or one. \square

Remark 7.11. *Note that we have shown that \mathfrak{b} -forking in one variable is caused by falling into some coset of G/n for some n . This may be seen as an analogue of the fact from stable theories that the beautiful pair associated to a one-based theory is again one-based.*

7.3. Small closure is a pregeometry.

Corollary 7.12. *If $\mathfrak{R}|_{\mathcal{L}}$ either is an ordered group with no additional structure or has a definable field structure, then the closure operator $\text{scl} : \mathcal{P}(R) \rightarrow \mathcal{P}(R)$ defines a pregeometry.*

Proof. Let $C \subset R$ and let $a, b \in R$ be such that $a \in \text{scl}(C \cup \{b\}) \setminus \text{scl}(C)$. Then $\text{tp}(a/C \cup \{b\})$ $\not\vdash$ -forks over C by either Corollary 7.5 or 7.10. By symmetry, $\text{tp}(b/C \cup \{a\})$ also $\not\vdash$ -forks over C , so by Corollary 3.5, $b \in \text{scl}(Ca)$. \square

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