

RATIONAL SOLUTIONS OF POLYNOMIAL-EXPONENTIAL EQUATIONS

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1. INTRODUCTION

Let $\mathbf{X} = (X_1, \dots, X_t)$ be a tuple of indeterminates. Fix $P_1, \dots, P_s \in \mathbb{C}[\mathbf{X}]$ and $\alpha_1, \dots, \alpha_s \in \mathbb{C}^t$. We consider the following equation

$$(\star) \quad \sum_{i=1}^s P_i(\mathbf{X}) \exp(\mathbf{X} \cdot \alpha_i) = 0,$$

where \exp denotes the usual complex exponentiation map and $\mathbf{X} \cdot \alpha_i$ is short for $X_1 \alpha_{i1} + \dots + X_t \alpha_{it}$.

A solution $\mathbf{a} = (a_1, \dots, a_t) \in \mathbb{C}^t$ of (\star) is said to be *nondegenerate* if

$$\sum_{i \in I} P_i(\mathbf{a}) \exp(\mathbf{a} \cdot \alpha_i) \neq 0$$

for every nonempty proper subset I of $\{1, \dots, s\}$.

Consider the \mathbb{Q} -vector space

$$V := \{\mathbf{q} \in \mathbb{Q}^t : \mathbf{q} \cdot \alpha_i = \mathbf{q} \cdot \alpha_{i'} \text{ for all } i, i'\}.$$

Fix a complement V' of V in \mathbb{Q}^t and let $\pi : \mathbb{Q}^t \rightarrow V$ and $\pi' : \mathbb{Q}^t \rightarrow V'$ denote the natural projections.

We give a description of nondegenerate rational solutions of (\star) as follows.

Theorem 1.1. *Given $P_1, \dots, P_s \in \mathbb{C}[\mathbf{X}]$ and $\alpha_1, \dots, \alpha_s \in \mathbb{C}^t$, there is $N \in \mathbb{N}^{>0}$ such that if $\mathbf{q} \in \mathbb{Q}^t$ is a nondegenerate solution of (\star) , then $\pi'(\mathbf{q}) \in (\frac{1}{N}\mathbb{Z})^t$.*

Note that searching for $\mathbf{n} = (n_1, \dots, n_t) \in \mathbb{Z}^t$ satisfying (\star) amounts to solving the classical polynomial-exponential equation

$$(\star\star) \quad \sum_{i=1}^s P_i(\mathbf{n}) \prod_{j=1}^t \beta_{ij}^{n_j} = 0,$$

where $\beta_{ij} = \exp(\alpha_{ij})$ ($i = 1, \dots, s, j = 1, \dots, t$). This kind of solutions were considered in [3]. For such an equation let

$$H = \{\mathbf{n} \in \mathbb{Z}^t : \prod_{j=1}^t \beta_{ij}^{n_j} = \prod_{j=1}^t \beta_{i'j}^{n_j} \text{ for every } i, i'\}.$$

Theorem 1.2. (Laurent [3]) (i) If P_i is constant for each i , then the set of non-degenerate solutions of $(\star\star)$ is a finite union of translates of H .

(ii) There are constants $a, b \in \mathbb{R}$ depending on the P_i 's and the β_{ij} 's such that if \mathbf{n} is a nondegenerate solution of $(\star\star)$, then there is $\mathbf{n}' \in H$ with

$$|\mathbf{n} - \mathbf{n}'| < a \log(|\mathbf{n}|) + b.$$

(Here and below $|(b_1, \dots, b_t)| := \max\{|b_1|, \dots, |b_t|\}$.)

This is a slightly weaker version of Théorème 1 of [3], because there the author considers not only nondegenerate solutions, but solutions that are 'maximal' with respect to a partition of $\{1, \dots, s\}$.

Combining Theorem 1.1 with Laurent's result, we get a finer description of rational solutions of (\star) .

Corollary 1.3. Given $P_1, \dots, P_s \in \mathbb{C}[\mathbf{X}]$ and $\alpha_1, \dots, \alpha_s \in \mathbb{C}^t$, there are $N \in \mathbb{N}^{>0}$ and $c_r, d_r \in \mathbb{R}$ for each $r \in V$ such that if $\mathbf{q} \in \mathbb{Q}^t$ is a nondegenerate solution of (\star) , then $\pi'(\mathbf{q}) \in (\frac{1}{N}\mathbb{Z})^t$ and there is $\mathbf{m} \in \mathbb{Z}^t$ satisfying $\mathbf{m} \cdot (\alpha_i - \alpha_{i'}) \in \mathbb{Z}2N\pi\sqrt{-1}$ for each i, i' such that

$$|\pi'(\mathbf{q}) - \frac{\mathbf{m}}{N}| < c_{\pi(\mathbf{q})} \log(|\pi'(\mathbf{q})|) + d_{\pi(\mathbf{q})}.$$

We also have the following finiteness result as a consequence of Corollary 1.3.

Corollary 1.4. Let $P_1, \dots, P_s \in \mathbb{C}[X]$ and $\alpha_1, \dots, \alpha_s \in \mathbb{C}^t$. Suppose that the set $\{\beta_{ij} : i = 1, \dots, s, j = 1, \dots, t\}$ is multiplicatively independent. Then there are only finitely many nondegenerate solutions $\mathbf{q} \in \mathbb{Q}^t$ of (\star) .

2. LINEAR RELATIONS IN MULTIPLICATIVE GROUPS

In this section we recall some notations and an earlier result that will be useful in the rest of the paper and we make the first reduction.

Let K be any field and Γ a subgroup of K^\times . We consider solutions in Γ of

$$(*) \quad \lambda_1 x_1 + \dots + \lambda_k x_k = 1,$$

where $\lambda_1, \dots, \lambda_k \in K$. We say that a solution $\gamma = (\gamma_1, \dots, \gamma_k)$ in Γ of $(*)$ is *non-degenerate* if $\sum_{i \in I} \lambda_i \gamma_i \neq 0$ for every nonempty proper subset I of $\{1, \dots, k\}$.

Definition 2.1. Let G be an abelian group, written multiplicatively. A subgroup H of G is *radical* (in G) if for each $n > 0$ and $g \in G$ we have $g \in H$ whenever $g^n \in H$.

Given $A \subseteq G$, we set $\langle A \rangle_G$ to be the smallest radical subgroup of G containing A . That is,

$$\langle A \rangle_G = \{g \in G \mid g^n \in [A]_G \text{ for some } n \in \mathbb{N}\}$$

where $[A]_G$ denotes the subgroup generated by A . When G is clear from the context, we will drop the subscripts and just write $\langle A \rangle$ and $[A]$.

Also in what follows, \mathbb{U} denotes the multiplicative group of roots of unity.

We use the following result from [1].

Lemma 2.2. *Let $E \subseteq F$ be fields such that $E \cap \mathbb{U} = F \cap \mathbb{U}$ and G be a radical subgroup of E^\times . Then for $\lambda_1, \dots, \lambda_n \in E^\times$, the equation $(*)$ has the same non-degenerate solutions in G as in $\langle G \rangle_{F^\times}$.*

In the rest of the paper, it will be more convenient to consider the following equation rather than (\star)

$$(\star\star) \quad \sum_{i=1}^s P_i(X) \exp(X \cdot \alpha'_i) = 0,$$

where $\alpha'_i = (\alpha_{ij} - \alpha_{1j})_{j=1, \dots, t}$. Note that \mathbf{q} is a nondegenerate solution of (\star) if and only if it is a nondegenerate solution of $(\star\star)$. Hence we do not lose any information by replacing α_i with α'_i . Let S denote the set of nondegenerate solutions $\mathbf{q} \in \mathbb{Q}^t$ of $(\star\star)$, and put $\beta'_{ij} = \exp(\alpha_{ij} - \alpha_{1j})$.

Let A be a finite set containing the coefficients of the P_i 's and the β'_{ij} 's and put $\Gamma = \langle A \rangle_{\mathbb{C}^\times}$. If $\mathbf{q} \in S$, then $P_1(\mathbf{q}) \neq 0$ and the tuple $(\exp(\mathbf{q} \cdot \alpha'_i) : i = 2, \dots, s) \in \Gamma^{s-1}$ is a non-degenerate solution of the linear equation

$$(**) \quad \frac{P_2(\mathbf{q})}{-P_1(\mathbf{q})} Y_2 + \dots + \frac{P_s(\mathbf{q})}{-P_1(\mathbf{q})} Y_s = 1.$$

Note that when \mathbf{q} varies in \mathbb{Q}^t with $P_1(\mathbf{q}) \neq 0$, the coefficients of this equation vary in the field $\mathbb{Q}(A)$.

Let $E := \mathbb{Q}(\mathbb{U} \cup A)$ and $G := \langle A \rangle_{E^\times}$. Now by taking \mathbb{C} in the place of F in Lemma 2.2, we see that all the possible solutions of the linear equation $(**)$ in Γ are in G .

Let G' be the complement of \mathbb{U} in G . Therefore if $\mathbf{q} \in S$, then there are roots of unity $\zeta_{\mathbf{q}2}, \dots, \zeta_{\mathbf{q}s}$ and $\eta_{\mathbf{q}2}, \dots, \eta_{\mathbf{q}s} \in G'$ such that

$$(***) \quad P_1(\mathbf{q}) + P_2(\mathbf{q})\eta_{\mathbf{q}2}\zeta_{\mathbf{q}2} + \dots + P_s(\mathbf{q})\eta_{\mathbf{q}s}\zeta_{\mathbf{q}s} = 0.$$

We elaborate on the relation between \mathbf{q} and the $\eta_{\mathbf{q}i}$'s later; we first conclude that there is a bound on the order of the $\zeta_{\mathbf{q}i}$'s.

3. ROOTS OF UNITY

We first remark the following easy observation, whose proof follows the lines of the proof of Lemme 4 of [3] (Note that Laurent considered only finitely generated \mathbb{Q} -algebras, however his result is deeper).

Lemma 3.1. *Let R be a subring of $\overline{\mathbb{Q}}[S]$, where S is a finite subset of \mathbb{C} . Suppose that b_1, \dots, b_q are elements of R and let q' be the linear dimension over $\overline{\mathbb{Q}}$ of \mathbf{b} . Then there are ring homomorphisms $\phi_1, \dots, \phi_{q'}$ from R to $\overline{\mathbb{Q}}$ fixing $k := R \cap \overline{\mathbb{Q}}$ such that for every $\alpha_1, \dots, \alpha_q \in k$ with $\alpha_1 b_1 + \dots + \alpha_q b_q \neq 0$ there is some $i \in \{1, \dots, q'\}$ with $\phi_i(\alpha_1 b_1 + \dots + \alpha_q b_q) \neq 0$.*

Proof. After changing R , we may assume that $q = q'$. It suffices then to find $\phi_i : R \rightarrow \overline{\mathbb{Q}}$ for each $i \in \{1, \dots, q\}$ fixing k such that the determinant

$$D_q := \begin{vmatrix} \phi_1(b_1) & \dots & \phi_1(b_q) \\ \vdots & & \vdots \\ \phi_q(b_1) & \dots & \phi_q(b_q) \end{vmatrix}$$

is nonzero.

We proceed by induction on q . For $q = 1$ by Nullstellensatz take a ring homomorphism $R[b_1^{-1}] \rightarrow \bar{\mathbb{Q}}$ that fixes k . Clearly, its restriction to R sends b_1 to some non-zero element.

Assume now that $\phi_1, \dots, \phi_{q-1}$ have been already constructed such that $D_{q-1} \neq 0$. Then the determinant

$$D'_q := \begin{vmatrix} \phi_1(b_1) & \cdots & \phi_1(b_q) \\ \vdots & & \vdots \\ \phi_{q-1}(b_1) & \cdots & \phi_{q-1}(b_q) \\ b_1 & \cdots & b_q \end{vmatrix}$$

is $\beta_1 b_1 + \cdots + \beta_q b_q$, where β_1, \dots, β_q are algebraic numbers. In particular, by induction, $\beta_q = D_{q-1} \neq 0$. Therefore, since we are assuming that the tuple \mathbf{b} is $\bar{\mathbb{Q}}$ -linearly independent, we conclude that $D'_q \neq 0$. Consider

$$R' := R[(D'_q)^{-1}].$$

Nullstellensatz implies that there is a ring morphism ϕ_q from R' to $\bar{\mathbb{Q}}$ fixing $k' := R' \cap \bar{\mathbb{Q}}$. Its restriction to R has the property that $\phi_q D'_q \neq 0$ which implies that $D_q \neq 0$. \square

In order to bound the degrees of the roots of unity appearing in (***) we need the following result.

Theorem 3.2. (Theorem 1 in [2])

Let F be a number field, a_0, a_1, \dots, a_k in F and ζ a root of unity of order Q such that $a_0 + \sum_{j=1}^k a_j \zeta^{n_j} = 0$ with $\gcd(Q, n_1, \dots, n_k) = 1$. Let $\delta = [F \cap \mathbb{Q}(\zeta) : \mathbb{Q}]$ and suppose that for any nonempty proper subset I of $\{0, 1, \dots, k\}$ the sum $\sum_{j \in I} a_j \zeta^{n_j} \neq 0$. Then for each prime p and $n > 0$, if $p^{n+1} | Q$, then $p^n | 2\delta$ and

$$k \geq \dim_F(F + F\zeta^{n_1} + \cdots + F\zeta^{n_k}) \geq 1 + \sum_{p|Q, p^2 \nmid Q} \left[\frac{p-1}{\gcd(\delta, p-1)} - 1 \right].$$

In particular, the order Q of ζ is bounded by a constant depending on k and δ (and therefore $[F : \mathbb{Q}]$).

We are ready to prove the following.

Lemma 3.3. *There is $T \in \mathbb{N}$ depending only on the coefficients of the P_i 's and the $\beta_{i,j}$'s such that if \mathbf{q} is a nondegenerate solution of (***) , then all the corresponding $\zeta_{\mathbf{q}i}$'s are of order less than T .*

Proof. Let

$$\zeta_{\mathbf{q}i} = \exp\left(\frac{l_{\mathbf{q}i} 2\pi\sqrt{-1}}{n_{\mathbf{q}i}}\right),$$

with $l_{\mathbf{q}i} < n_{\mathbf{q}i}$ and $\gcd(l_{\mathbf{q}i}, n_{\mathbf{q}i}) = 1$. We need to find a bound T on the n_i 's independent of \mathbf{q} .

Let R be the $\bar{\mathbb{Q}}$ -algebra generated by elements of A and their inverses. Using Lemma 3.1 with appropriate b_1, \dots, b_q , choose by Lemma 3.1 some specialization ϕ such that

$$\phi(P_1(\mathbf{q})) \neq 0.$$

The homomorphism ϕ transforms (***) into a non-trivial relation

$$\sum_{i=1}^s \phi(P_i(\mathbf{q}))\phi(\eta_{\mathbf{q}i})\zeta_{\mathbf{q}i} = 0.$$

So we have a relation

$$\sum_{j \in I} a_j \zeta^j = 0,$$

where $\zeta = \exp(\frac{2\pi\sqrt{-1}}{N})$ with $N = \text{lcm}(n_{\mathbf{q}i} : i = 1, \dots, s)$ and the a_j 's are algebraic numbers depending on \mathbf{q} and not all zero.

For our purposes we may assume that no subsum is 0. Then Theorem 3.2 gives that T depends only on the degree of F and $|I|$ and not on \mathbf{q} . \square

4. PROOF OF THEOREM 1.1

In order to finish the proof of Theorem 1.1, we need the following Kummer theoretic result from [6].

Proposition 4.1. (*Bays-Zilber*) *Let L be a finitely generated extension of $\mathbb{Q}(\mathbb{U})$. Then the quotient group L^\times/\mathbb{U} is a free abelian group.*

It follows from this proposition that the group G' from Section 2 is finitely generated. Indeed G' can be considered as a subgroup of E^\times/\mathbb{U} , which is a free abelian group by the proposition and being of finite rank G' is actually finitely generated. Then by using Lemma 3.3, if $\mathbf{q} \in S$, then $(\mathbf{q} \cdot \alpha'_2, \dots, \mathbf{q} \cdot \alpha'_s)$ is in a finitely generated subgroup A of

$$W := \{(\mathbf{q} \cdot \alpha'_2, \dots, \mathbf{q} \cdot \alpha'_s) : \mathbf{q} \in \mathbb{Q}^t\}.$$

Note that W is a \mathbb{Q} -linear subspace of $\mathbb{C}^{(s-1)}$ and there is a natural surjective linear map $\phi : \mathbb{Q}^t \rightarrow W$ taking \mathbf{q} to $(\mathbf{q} \cdot \alpha'_2, \dots, \mathbf{q} \cdot \alpha'_s)$. Then the kernel of ϕ is V from the Introduction.

The preimage of A under ϕ is of the form $V \oplus B$ where B is a finitely generated subgroup of V' . Note that the rank of B is at most the dimension of V' .

Let $B = \mathbb{Z}\mathbf{r}_1 \oplus \dots \oplus \mathbb{Z}\mathbf{r}_d$, where $\mathbf{r}_1, \dots, \mathbf{r}_d \in \mathbb{Q}^t$. For $k = 1, \dots, d$, write \mathbf{r}_k as $(\frac{a_{k1}}{b_{k1}}, \dots, \frac{a_{kt}}{b_{kt}})$, where $a_{kj} \in \mathbb{Z}$, $b_{kj} \in \mathbb{N}^{>0}$ with $\text{gcd}(a_{jk}, b_{jk}) = 1$ for each $j = 1, \dots, t$.

Now let

$$N := \text{lcm}(b_{kj} : j = 1, \dots, t, k = 1, \dots, d).$$

Then $B \subseteq (\frac{1}{N}\mathbb{Z})^t$. Hence if $\mathbf{q} \in S$, then $\pi'(\mathbf{q}) \in (\frac{1}{N}\mathbb{Z})^t$, finishing the proof of Theorem 1.1.

5. FINAL REMARKS

The motivation for this work is to answer the following question affirmatively.

Question. Assume *Schanuel's conjecture*. Is it correct that for every irreducible polynomial $p(X, Y) \in \mathbb{C}[X, Y]$ in which both X and Y appear, there is a generic point of the form $(\alpha, \exp(\alpha))$ on the complex curve defined by p ?

Schanuel's conjecture is the statement that for every \mathbb{Q} -linearly independent complex numbers $\alpha_1, \dots, \alpha_n$, we have

$$\text{trdeg}_{\mathbb{Q}} \mathbb{Q}(\alpha_1, \dots, \alpha_n, \exp(\alpha_1), \dots, \exp(\alpha_n)) \geq n.$$

The question above stems from the model theoretic study of the field of complex numbers expanded by the usual exponential function, (\mathbb{C}, \exp) . In [5], Zilber gives an axiomatization of a first order theory and conjectures that (\mathbb{C}, \exp) is a model of that theory. Since one of the axioms is Schanuel's conjecture, it seems like Zilber's conjecture is out of reach. However, one can try to reduce it to Schanuel's conjecture. An affirmative answer to the question above would be a first step for such a purpose.

To find such a generic point, it is sufficient to show that for each subfield K of \mathbb{C} of finite transcendence degree, there are only finitely many $\alpha \in K$ such that $p(\alpha, \exp(\alpha)) = 0$ (see [4] for an explanation of this). So let K be such a subfield. Using Schanuel's conjecture one can conclude that $\alpha \in K$ with $p(\alpha, \exp(\alpha)) = 0$ lie in a \mathbb{Q} -linear subspace of K of finite dimension. Then the question reduces to understanding the rational solutions of certain polynomial-exponential equations. Unfortunately, the description we have in this article does not imply that there are only finitely many such solutions. However, it is still possible that restricting to the kind of equations that occur as above we have a better description of rational solutions.

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