Algebra II

Homework 5 - Due April 13, 2026, 11:00

1. Let k be a field and let $f, g \in k[X]$. Define the *resultant* of f and g to be

$$\mathrm{Res}(f,g) = \prod_{i=1}^{n} g(\alpha_i),$$

where $\alpha_1, \ldots, \alpha_n \in \overline{k}$ are the roots of f. Also recall that the *discriminant* of f is $\Delta(f) = \prod_{i \neq j}(\alpha_i = \alpha_j)$.

   (a) Show that $\mathrm{Res}(f,g) = 0$ if and only if f and g have a common root.

   (b) Show that $\mathrm{Res}(f,f') = (-1)^{\binom{n}{2}}\Delta(f)$.

   (c) Let $n > 0$ be odd and let $\zeta \in \mathbb{C}$ be a primitive n^{th}-root of unity. Show that there is $\alpha \in \mathbb{Q}(\zeta)$ such that $\alpha^2 = (-1)^{\binom{n}{2}} n$.

2. Let F be a field, and T an indeterminate. Let $f, g \in F[T]$ be relatively prime. Suppose also that it is not the case that both f and g constant and that $g \neq 0$. Show that

$$[F(T) : F(\frac{f}{g})] = \max\{\deg f, \deg g\}.$$

3. Find an irreducible polynomial of degree 5 over $\mathbb{Q}$ that is solvable by radicals.

4. Let $K = \mathbb{C}(T)$ where T is an indeterminate, and let $\zeta \in \mathbb{C}$ be a primitive third root of unity. Also let σ, τ be the automorphisms of K over $\mathbb{C}$ with $\sigma(T) = \zeta T$ and $\tau(T) = T^{-1}$. Find $f \in K$ such that the fixed field of the group of automorphisms of K generated by σ and τ is $\mathbb{C}(f)$.