

Algebra II

HOMEWORK 4 - DUE MARCH 25, 2026, 11:00

Let G be a group. A G -module is an abelian group $(M, +)$ with an action of G respecting $+$. For a G -module M , a 1-cocycle is a subset of M of the form $\{\alpha_\sigma \in M : \sigma \in G\}$ such that

$$\alpha_\sigma + \sigma\alpha_\tau = \alpha_{\sigma\tau} \quad \text{for all } \sigma, \tau \in G.$$

- (a) Let K be a field and $G \leq \text{Aut}(K)$. How can we interpret $(K, +)$ and (K^\times, \cdot) as G -modules?
- (b) Let $C = \{\alpha_\sigma : \sigma \in G\}$ be a 1-cocycle. Given $\sigma \in G$, calculate α_{σ^n} in terms of α_σ . What is α_1 ?
- (c) Show that if G is a cyclic group generated by σ , then there is a one-to-one correspondence between 1-cocycles and elements $\alpha \in M$ satisfying

$$\alpha + \sigma(\alpha) + \sigma^2(\alpha) + \cdots + \sigma^{n-1}(\alpha) = 0.$$

- (d) Show that 1-cocycles form a group with the addition:

$$\{\alpha_\sigma : \sigma \in G\} + \{\beta_\sigma : \sigma \in G\} = \{\alpha_\sigma + \beta_\sigma : \sigma \in G\}.$$

Denote this group as $Z^1(G, M)$.

- (e) A 1-coboundary is a subset of M of the form $\{\alpha_\sigma \in M : \sigma \in G\}$ such that there is $\beta \in M$ with the property

$$\alpha_\sigma = \sigma(\beta) - \beta \quad \text{for every } \sigma \in G.$$

Show that the set of 1-coboundaries form a subgroup of $Z^1(G, M)$.

Denote this subgroup by $B^1(G, M)$ and the quotient $Z^1(G, M)/B^1(G, M)$ is called the *first cohomology group* of G in M , denoted by $H^1(G, M)$.

- (f) Let $K|k$ be a finite Galois extension with Galois group G . Show that $H^1(G, K^\times)$ is trivial. (Hint: Linear independence of characters.)
- (g) Deduce Hilbert's 90 from the previous part.