

## Algebra II

### HOMEWORK 3 - DUE MARCH 11, 2026, 11:00

1. Find the Galois groups of the following polynomials over the indicated fields.

- (a)  $X^4 + 3X^2 + 7X + 4$  over  $\mathbb{Q}$
- (b)  $X^3 - 5$  over  $\mathbb{Q}(\sqrt{-3})$
- (c)  $X^3 + T^2X - T^3$  over  $\mathbb{C}(T)$

2. Let  $K$  be a field and let

$$L = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in K \right\}.$$

- (a) Show that  $L$  becomes a field with matrix addition and multiplication if and only if  $-1$  is not a square in  $K$ .
- (b) If  $-1$  is not a square in  $K$ , then  $K$  embeds into  $L$ .
- (c) Construct a field with 9 elements using the previous parts.
- (d) Let

$$E = \left\{ \begin{pmatrix} a & b \\ -b & a - b \end{pmatrix} : a, b \in K \right\}.$$

Find a condition that is equivalent to  $E$  being a field with matrix addition and multiplication.

3. (a) Find a finite extension  $K|k$  that has infinitely many intermediate fields.
- (b) Find an infinite extension  $K|k$  such that every element has degree at most  $N$  for some  $N \in \mathbb{N}$  depending only on the extension.