Algebra II

Homework 2 - Due March 2, 2026, 11:00

1. Let $K = k(\alpha_1, \ldots, \alpha_n)$ be an algebraic extension.

   (a) Suppose that for every i, all the conjugates of α_i are in K. Show that $K|k$ is normal.

   (b) Suppose that α_i is separable over k for every i. Show that $K|k$ is separable.

2. Let K be a finite extension of $\mathbb{F}_p$ of degree n and let $\phi : K \to K$ be the Frobenius map; that is $\phi(\alpha) = \alpha^p$ for $\alpha \in K$.

   a. Show that ϕ is an $\mathbb{F}_p$-linear map.

   b. Show that the order of ϕ in the group $\mathrm{End}_{\mathbb{F}_p}(K)$ is n.

   c. Explain how K becomes an $\mathbb{F}_p[T]$-module via ϕ.

   d. Find the Rational Canonical form of K (as an $\mathbb{F}_p[T]$-module).

3. Let K be a field of characteristic $p > 0$.

   (a) Let α be an element of K that is not a p^{th} power (in K). Show that $X^{p^n} - \alpha$ is irreducible for every $n > 0$.

   (b) Let $\beta \in \overline{K}$. Show that β is separable over K if and only if $K(\beta) = K(\beta^{p^n})$ for every $n > 0$.