DEFINABLE SETS IN MANN PAIRS

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Abstract. Consider structures $(\Omega, \mathbf{k}, \Gamma)$ where Ω is an algebraically closed field of characteristic zero, $\mathbf{k}$ is a subfield, and Γ is a subgroup of the multiplicative group of Ω. Certain pairs $(\mathbf{k}, \Gamma)$ have been singled out as Mann pairs in [3]. We give new examples of such Mann pairs, and for a Mann pair $(\mathbf{k}, \Gamma)$ we axiomatize the first-order theory of $(\Omega, \mathbf{k}, \Gamma)$ in a cleaner way than in [3], and, as the main result of the paper, we characterize the subsets of Ω^n that are definable in $(\Omega, \mathbf{k}, \Gamma)$.

1. Introduction

This paper is a sequel to [3]. We let Ω be an algebraically closed (ambient) field, $\mathbf{k}$ a subfield of Ω, and Γ a subgroup of $\Omega^\times$. Also m, n range over $\mathbb{N} = \{0, 1, 2, \dots\}$ and for $a \in \Omega$ and $\vec{s} = (s_1, \dots, s_n) \in \Omega^n$ we put $a\vec{s} := (as_1, \dots, as_n) \in \Omega^n$. For $n \geq 1$ we set

$$\Gamma^{[n]} := \{\gamma^n : \gamma \in \Gamma\} \qquad \text{(a subgroup of } \Gamma\text{)}.$$

Other notations are explained as needed.

Let $n \geq 2$ and $a_1, \dots, a_n \in \Omega$. A *nondegenerate solution* of the equation

$$a_1x_1 + \cdots + a_nx_n = 0$$

is a tuple $\vec{s} = (s_1, \dots, s_n) \in (\Omega^\times)^n$ such that $a_1s_1 + \cdots + a_ns_n = 0$ and $\sum_{i \in I} a_is_i \neq 0$ for all proper nonempty subsets I of $\{1, \dots, n\}$; note that then $a_1, \dots, a_n \neq 0$, and that $a\vec{s}$ for $a \in \Omega^\times$ is also a nondegenerate solution of the same equation, and so for most purposes we can normalize to $s_n = 1$.

Let $n \geq 2$ and define $\Gamma(\mathbf{k}, n)$ to be the set of all $\vec{\gamma} = (\gamma_1, \dots, \gamma_n) \in \Gamma^n$ such that $\gamma_n = 1$ and $\vec{\gamma}$ is a nondegenerate solution of some equation

$$a_1x_1 + \cdots + a_nx_n = 0$$

with $a_1, \dots, a_n \in \mathbf{k}$. Recall from [3] that $(\mathbf{k}, \Gamma)$ is a *Mann pair* if and only if $\Gamma(\mathbf{k}, n)$ is finite for all $n \geq 2$.[1] One example of a Mann pair is $(\mathbb{Q}, \mathbb{U})$ within the ambient field $\mathbb{C}$, where $\mathbb{U}$ is the subgroup of $\mathbb{C}^\times$ consisting of all the roots of unity (see [7] for a proof of this fact). Also, if Ω is of characteristic zero, $\mathbf{k}$ is algebraically closed, $\mathbf{k}^\times \cap \Gamma = \{1\}$ and Γ is of finite rank (that is, Γ has a finitely generated subgroup Γ_0 such that Γ/Γ_0 is a torsion group), then $(\mathbf{k}, \Gamma)$ is a Mann pair; see Theorem 1.1 of [3].

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[1] This is not quite the definition of Mann pair in [3], but is equivalent to it.

Let $n \geq 2$ and $a_1, \ldots, a_n \in \mathbf{k}^\times$. A solution $\vec{s} = (s_1, \ldots, s_n) \in (\Omega^\times)^n$ of

$$(*) \qquad a_1 x + \cdots + a_n x_n = 0$$

is said to be *primitive over* $\mathbf{k}$ if $(s_i)_{i \in I}$ is linearly independent over $\mathbf{k}$ for every nonempty proper subset I of $\{1, \ldots, n\}$. So a primitive solution of $(*)$ over $\mathbf{k}$ is in particular a nondegenerate solution of $(*)$.

For $n \geq 2$, let $\Gamma(\mathbf{k}, n)^{pr}$ be the set of $\vec{\gamma} \in \Gamma^n$ such that $\gamma_n = 1$ and $\vec{\gamma}$ is a primitive solution of $(*)$ over $\mathbf{k}$ for some $a_1, \ldots, a_n \in \mathbf{k}^\times$; in particular, $\Gamma(\mathbf{k}, n)^{pr} \subseteq \Gamma(\mathbf{k}, n)$. In Section 3 we axiomatize the first-order theory of $(\Omega, \mathbf{k}, \Gamma)$ when $(\mathbf{k}, \Gamma)$ is a Mann pair. More precisely, let $\mathcal{L}$ be the language of rings augmented by two distinct unary relation symbols, and let T be the $\mathcal{L}$-theory whose models are the structures $(\Omega, \mathbf{k}, \Gamma)$.

Theorem 1.1. *Suppose* $\mathbf{k}$ *is infinite,* $(\mathbf{k}, \Gamma)$ *is a Mann pair with* $[\Omega : \mathbf{k}] > 2$*, and* $(\Omega', \mathbf{k}', \Gamma')$ *is a model of* T *with* $(\Omega, \mathbf{k}, \Gamma)$ *as a substructure such that* $[\Omega' : \mathbf{k}'] > 2$*, and* $\Gamma'(\mathbf{k}', n)^{pr} = \Gamma(\mathbf{k}, n)^{pr}$ *for every* $n \geq 2$*. Then*

$$(\Omega', \mathbf{k}', \Gamma') \equiv_{\mathbf{k} \cup \Gamma} (\Omega, \mathbf{k}, \Gamma) \iff \mathbf{k} \preceq \mathbf{k}' \text{ and } \Gamma \preceq \Gamma'.$$

This improves on related results from [3] in not involving a choice of finite subset of Γ^n for $n = 2, 3 \ldots$, nor a choice of basis for the $\mathbf{k}$-linear spaces attached to the elements of these finite sets.

For Mann pairs $(\mathbf{k}, \Gamma)$ the subsets of $\mathbf{k}^m \times \Gamma^n$ that are definable in $(\Omega, \mathbf{k}, \Gamma)$ are determined in Proposition 7.2 of [3], but in the present paper we wish to describe more generally the subsets of Ω^m definable in $(\Omega, \mathbf{k}, \Gamma)$:

Theorem 1.2. *Suppose* $(\mathbf{k}, \Gamma)$ *is a Mann pair,* $\mathbf{k}$ *is algebraically closed,* $\mathbf{k} \neq \Omega$*, and* $\Gamma / \Gamma^{[p]}$ *is finite for each prime* p*. Then a subset of* Ω^m *is definable in* $(\Omega, \mathbf{k}, \Gamma)$ *if and only if it is a boolean combination of subsets of* Ω^m *of the form*

$$\bigcup_{\vec{a} \in \mathbf{k}^d} \bigcup_{\vec{\gamma} \in \Gamma^e} X(\vec{a}, \vec{\gamma}), \qquad (d, e \in \mathbb{N})$$

where $X \subseteq \Omega^{d+e+m}$ *is definable in the field* Ω *and* $X(\vec{a}, \vec{\gamma})$ *is the set of all* $\vec{s} \in \Omega^m$ *such that* $(\vec{a}, \vec{\gamma}, \vec{s}) \in X$.

In other words, $(\Omega, \mathbf{k}, \Gamma)$ eliminates quantifiers down to existential formulas with quantifiers ranging only over $\mathbf{k}$ and Γ.

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2. Some new Mann pairs

We indicate here some natural Mann pairs that we noticed recently. The first is an "easy" Mann pair in the sense of Section 2.2 of [3], and originates from the following classical result due to E. Borel [1], p. 387:

Let $f_1, \ldots, f_n$ *with* $n \geq 1$ *be entire functions of one complex variable* z *that are linearly independent over* $\mathbb{C}$*, and suppose each* f_j *as well as* $f_1 + \cdots + f_n$

has only finitely many zeros. Then there is an entire function h and there are polynomials $p_j \in \mathbb{C}[z]$ such that $f_j = p_j e^h$ for $j = 1, \dots, n$.

Consider now inside the field of meromorphic functions on the complex plane $\mathbb{C}$ the subfield $\mathbf{k} = \mathbb{C}(z)$ of rational functions, and the multiplicative group $\Gamma := \{e^h : \ h \text{ is entire and } h(0) = 0\}$. It is easy to see that Borel's theorem is equivalent to the proposition that $(\mathbf{k}, \Gamma)$ is an easy Mann pair. The condition $h(0) = 0$ is just a normalization to arrange that Γ is torsion-free and $\mathbf{k}^\times \cap \Gamma = \{1\}$.

Next, let $\mathbf{k}$ be any field of characteristic 0. Then the generalized power series field $K := \mathbf{k}((t^{\mathbb{Q}}))$ in the variable t comes with a valuation $v : K^\times \to \mathbb{Q}$, and a derivation d/dt on K with constant field $\mathbf{k}$. Let

$$\mathcal{O} := \{f \in K : \ vf \geq 0\}, \quad \mathfrak{m} := \{f \in K : \ vf > 0\}$$

be the valuation ring of v and its maximal ideal. The exponential function

$$f \mapsto \exp(f) := \sum_{n=0}^{\infty} \frac{f^n}{n!} \ : \ \mathfrak{m} \to 1 + \mathfrak{m},$$

is an isomorphism of the additive group $\mathfrak{m}$ onto the subgroup $1 + \mathfrak{m}$ of $K^\times$, and $\exp(f)' = f' \exp(f)$ for $f \in \mathfrak{m}$, with $g' := dg/dt$.

Proposition 2.1. *Let L be a finite dimensional subspace of the $\mathbf{k}$-linear space $\mathfrak{m}$, and put $\Gamma := \exp L$. Then $(\mathbf{k}, \Gamma)$ is a Mann pair.*

Proof. We can basically repeat the proof at the end of Section 6 of [3], which is based on Corollary 2.7 in [5]: assume without loss that $\mathbf{k}$ is algebraically closed and extend the differential field K (with derivation d/dt) to a differentially closed field Ω with constant field $\mathbf{k}$. Consider the logarithmic-derivative map $\mathrm{ld} : \Omega^\times \to \Omega$, $\mathrm{ld}(x) := x'/x$. Take $r \in \mathbb{N}$ and $f_1, \dots, f_r \in \mathfrak{m}$ with $L = \mathbf{k} \cdot f_1 + \cdots + \mathbf{k} \cdot f_r$ and put $\gamma_i := \exp f_i$. Then

$$\mathrm{ld}(\Gamma) = \mathbf{k} \cdot f_1' + \cdots + \mathbf{k} \cdot f_r' = \mathbf{k} \cdot \mathrm{ld}(\gamma_1) + \cdots + \mathbf{k} \cdot \mathrm{ld}(\gamma_r),$$

and from here on the proof at the end of Section 6 of [3] goes through word for word. □

This proposition and its proof go through for some other differential fields with a (partial) exponential function, like the field $\mathbb{R}[[[x]]]$ of transseries.

3. AXIOMATIZING $(\Omega, \mathbf{k}, \Gamma)$

Inspired by Section 6 of [2] we first show that the solutions in Γ of equations

$$a_1 x_1 + \cdots + a_n x_n = 0, \quad (a_1, \dots, a_n \in \mathbf{k}^\times)$$

are generated in a certain way by its primitive solutions in Γ.

3.1. **Linear considerations.**

Below, $\vec{s} = (s_1, \dots, s_n) \in (\Omega^\times)^n$, $n \geq 2$, and $I \subseteq \{1, \dots, n\}$. We say that I is *$\vec{s}$-minimal over* $\mathbf{k}$ if the tuple $(s_i)_{i \in I}$ is linearly dependent over $\mathbf{k}$ (hence $|I| \geq 2$), and for each proper subset J of I the tuple $(s_j)_{j \in J}$ is linearly independent over $\mathbf{k}$.

Suppose I is $\vec{s}$-minimal over $\mathbf{k}$. Then there is a tuple $(a_i)_{i \in I}$ with all $a_i \in \mathbf{k}$ and some $a_i \neq 0$ such that $\sum_{i \in I} a_i s_i = 0$; such a tuple (a_i) is unique up to multiplication by a nonzero scalar from $\mathbf{k}$ and has $a_i \neq 0$ for all i.

Define $V(\vec{s})$ to be the $\mathbf{k}$-linear subspace of $\mathbf{k}^n$ consisting of all $\vec{a} = (a_1, \dots, a_n)$ such that $a_1 s_1 + \cdots + a_n s_n = 0$. If I is $\vec{s}$-minimal over $\mathbf{k}$, then $V\big((s_i)_{i \in I}\big)$ is a one-dimensional subspace of $\mathbf{k}^I$. For each I that is $\vec{s}$-minimal over $\mathbf{k}$, fix an element a^I of $\mathbf{k}^n$ such that $(a^I_i)_{i \in I}$ generates $V\big((s_i)_{i \in I}\big)$ and $a^I_j = 0$ for $j \notin I$. The next two lemmas are reformulations of Lemmas 4 and 5 in [2].

Lemma 3.1. *The $\mathbf{k}$-linear space $V(\vec{s})$ is generated by the a^I for which I is $\vec{s}$-minimal over $\mathbf{k}$.*

For $J \subseteq \{1, \dots, n\}$ we put $J^c := \{1, \dots, n\} \setminus J$.

Lemma 3.2. *Let $\vec{a} \in V(\vec{s})$, $J \subseteq \{1, \dots, n\}$, and $\sum_{j \in J} a_j s_j \neq 0$. Then there is an I that is $\vec{s}$-minimal over $\mathbf{k}$ and meets both J and J^c.*

The direction (1) ⇒ (2) of the next lemma is essentially Lemma 6 of [2], but we also need (2) ⇒ (1), so we give a complete proof.

Lemma 3.3. *Assume $\mathbf{k}$ is infinite. Then the following are equivalent:*

1. *$\vec{s}$ is a nondegenerate solution of some equation $a_1 x_1 + \cdots + a_n x_n = 0$ with $a_1, \dots, a_n \in \mathbf{k}$;*
2. *$\{1, \dots, n\}$ can be covered by subsets $I_1, \dots, I_m$ that are $\vec{s}$-minimal over $\mathbf{k}$, such that for every nonempty proper subset J of $\{1, \dots, n\}$ some I_p with $p \in \{1, \dots, m\}$ meets both J and J^c.*

Proof. To show that (1) implies (2), let $a_1, \dots, a_n \in \mathbf{k}^\times$ be such that $\vec{s}$ is a nondegenerate solution of $a_1 x_1 + \cdots + a_n x_n = 0$. Take an $I_1 \subseteq \{1, \dots, n\}$ that is $\vec{s}$-mininal over $\mathbf{k}$. Note that if $I_1 = \{1, \dots, n\}$, then (2) holds with $m = 1$. Suppose $I_1, \dots, I_m \subseteq \{1, \dots, n\}$ with $m \geq 1$ are $\vec{s}$-minimal over $\mathbf{k}$, we have a strictly increasing chain

$$I_1 \subseteq I_1 \cup I_2 \subseteq \cdots \subseteq I_1 \cup \cdots \cup I_m,$$

and for every nonempty proper subset J of $I := I_1 \cup \cdots \cup I_m$ some I_p with $p \in \{1, \dots, m\}$ meets both J and J^c. If $I = \{1, \dots, n\}$, then (2) holds. Assume $I \neq \{1, \dots, n\}$. Then $\sum_{i \in I} a_i s_i \neq 0$, so by Lemma 3.2 we have an $I_{m+1} \subseteq \{1, \dots, n\}$ that is $\vec{s}$-minimal over $\mathbf{k}$ and meets both I and I^c. It follows easily that then for every nonempty proper subset J of $I \cup I_{m+1}$ some I_p with $p \in \{1, \dots, m+1\}$ meets both J and J^c. So in a finite number of steps we obtain a covering as in (2).

To prove the converse, assume (2). Take $I_1, \dots, I_m$ as in (2), and for $p = 1, \dots, m$, take $a_{pj} \in \mathbf{k}$, $j = 1, \dots, n$, such that

$$\sum_{j=1}^{n} a_{pj} s_j = 0,\ a_{pj} = 0 \text{ for } j \notin I_p,\ a_{pj} \neq 0 \text{ for } j \in I_p.$$

With $x_1, \dots, x_m \in \mathbf{k}$ we have

$$\begin{aligned} 0 &= x_1 \sum_{j=1}^{n} a_{1j} s_j + \cdots + x_m \sum_{j} a_{mj} s_j \\ &= (\sum_{p=1}^{m} a_{p1} x_i) s_1 + \cdots + (\sum_{p=1}^{m} a_{pn} x_p) s_n \end{aligned}$$

Thus it suffices to find $x_1, \dots, x_m \in \mathbf{k}$ such that for each nonempty proper subset J of $\{1, \dots, n\}$ we have $\sum_{j \in J} (\sum_{p=1}^{m} a_{pj} x_i) s_j \neq 0$, that is,

$$(\sum_{j \in J} a_{1j} s_j) x_1 + \cdots + (\sum_{j \in J} a_{mj} s_j) x_m \neq 0.$$

For each nonempty proper subset J of $\{1, \dots, n\}$ we take $p \in \{1, \dots, m\}$ such that I_p meets both J and J^c, and so $\sum_{j \in J} a_{pj} s_j \neq 0$. Since $\mathbf{k}$ is infinite, this yields $x_1, \dots, x_m \in \mathbf{k}$ as desired. □

Lemma 3.3 says basically how $\Gamma(\mathbf{k}, n)$ is determined by the sets $\Gamma(\mathbf{k}, m)^{pr}$ with $m = 2, \dots, n$. Here are some consequences:

Corollary 3.4. $(\mathbf{k}, \Gamma)$ *is a Mann pair iff* $\Gamma(\mathbf{k}, n)^{pr}$ *is finite for all* $n \geq 2$.

Proof. Let $n \geq 2$ be given and assume $\Gamma(\mathbf{k}, m)^{pr}$ is finite for $m = 2, \dots, n$. We shall derive that $\Gamma(\mathbf{k}, n)$ is finite. Let $\vec{\gamma} \in \Gamma(\mathbf{k}, n)$. The proof of the direction (1) ⇒ (2) of Lemma 3.3 does not use that $\mathbf{k}$ is infinite, so we have a covering of $\{1, \dots, n\}$ by subsets $I_1, \dots, I_m$ that are $\vec{\gamma}$-minimal over $\mathbf{k}$, such that for every nonempty proper subset J of $\{1, \dots, n\}$ some I_p meets both J and J^c. By renumbering the I's we arrange that $n \in I_1$, and since $\gamma_n = 1$ this leaves only finitely many possibilities for $(\gamma_i)_{i \in I_1}$. If $I_1 = \{1, \dots, n\}$ we are done. Otherwise, we can assume that I_2 meets both I_1 and I_1^c. Taking $i_1 \in I_1 \cap I_2$ we have only finitely many possibilities for γ_{i_1}, and so there are only finitely many possibilities for $(\gamma_i)_{i \in I_2}$ and thus for $(\gamma_i)_{i \in I_1 \cup I_2}$. If $I_1 \cup I_2 = \{1, \dots, n\}$ we are done, and otherwise we continue as above. □

Corollary 3.5. *Suppose* $\mathbf{k}$ *is infinite and* $K \supseteq \mathbf{k}$ *is a subfield of* Ω *such that* $\Gamma(\mathbf{k}, n)^{pr} = \Gamma(K, n)^{pr}$ *for all* $n \geq 2$. *Then* $\Gamma(\mathbf{k}, n) = \Gamma(K, n)$ *for all* $n \geq 2$.

Proof. Let $n \geq 2$ and $\vec{\gamma} \in \Gamma(\mathbf{k}, n)$. Then the direction (1) ⇒ (2) of Lemma 3.3 yields a covering of $\{1, \dots, n\}$ by subsets $I_1, \dots, I_m$ that are $\vec{\gamma}$-minimal over $\mathbf{k}$, such that for every nonempty proper subset J of $\{1, \dots, n\}$ some I_p with $p \in \{1, \dots, m\}$ meets both J and J^c. Then $I_1, \dots, I_m$ are also $\vec{\gamma}$-minimal over K, so by the direction (2) ⇒ (1) of Lemma 3.3 we have $\vec{\gamma} \in \Gamma(K, n)$. □

This gives an improvement of (4) in Section 5 of [3] for infinite $\mathbf{k}$:

Corollary 3.6. *Suppose* $\mathbf{k}$ *is infinite and* $K \supseteq \mathbf{k}$ *is subfield of* Ω *that is linearly disjoint from* $\mathbf{k}(\Gamma)$ *over* $\mathbf{k}$. *Then* $\Gamma(\mathbf{k}, n) = \Gamma(K, n)$ *for all* $n \geq 2$.

Proof. The linear disjointness assumption yields $\Gamma(\mathbf{k}, n)^{pr} = \Gamma(K, n)^{pr}$ for all $n \geq 2$. Now use Corollary 3.5. $\square$

3.2. **Elementary classification.** Consider a model $(\Omega_0, \mathbf{k}_0, \Gamma_0)$ of T such that $\mathbf{k}_0$ is infinite and $(\mathbf{k}_0, \Gamma_0)$ is a Mann pair. We now have Theorem 1.1 in the following stronger form.

Theorem 3.7. *Let* $(\Omega_1, \mathbf{k}_1, \Gamma_1)$ *and* $(\Omega_2, \mathbf{k}_2, \Gamma_2)$ *be models of* T *such that*

(1) $[\Omega_1 : \mathbf{k}_1] > 2$ *and* $[\Omega_2 : \mathbf{k}_2] > 2$;
(2) $(\Omega_0, \mathbf{k}_0, \Gamma_0) \subseteq (\Omega_1, \mathbf{k}_1, \Gamma_1)$ *and* $(\Omega_0, \mathbf{k}_0, \Gamma_0) \subseteq (\Omega_2, \mathbf{k}_2, \Gamma_2)$;
(3) $\Gamma_1(\mathbf{k}_1, n)^{pr} = \Gamma_2(\mathbf{k}_2, n)^{pr} = \Gamma_0(\mathbf{k}_0, n)^{pr}$ *for all* $n \geq 2$.

Then: $\quad (\Omega_1, \mathbf{k}_1, \Gamma_1) \equiv_{\mathbf{k}_0 \cup \Gamma_0} (\Omega_2, \mathbf{k}_2, \Gamma_2) \iff \mathbf{k}_1 \equiv_{\mathbf{k}_0} \mathbf{k}_2$ *and* $\Gamma_1 \equiv_{\Gamma_0} \Gamma_2$.

Proof. It follows easily from Lemma 3.1 and Corollary 3.5 that $(\Omega_i, \mathbf{k}_i, \Gamma_i)$ satisfies the Mann axioms of $(\Omega_0, \mathbf{k}_0, \Gamma_0)$ for $i = 1, 2$, in the sense of [3]. It remains to use Theorem 8.4 in [3]. $\square$

For algebraically closed $\mathbf{k}_i$ this gives:

Corollary 3.8. *Let* $(\Omega_1, \mathbf{k}_1, \Gamma_1)$ *and* $(\Omega_2, \mathbf{k}_2, \Gamma_2)$ *be models of* T *such that*

(1) $\mathbf{k}_1$ *and* $\mathbf{k}_2$ *are algebraically closed,* $\mathbf{k}_1 \neq \Omega_1$, $\mathbf{k}_2 \neq \Omega_2$;
(2) $(\Omega_0, \mathbf{k}_0, \Gamma_0) \subseteq (\Omega_1, \mathbf{k}_1, \Gamma_1)$ *and* $(\Omega_0, \mathbf{k}_0, \Gamma_0) \subseteq (\Omega_2, \mathbf{k}_2, \Gamma_2)$;
(3) $\Gamma_1(\mathbf{k}_1, n)^{pr} = \Gamma_2(\mathbf{k}_2, n)^{pr} = \Gamma_0(\mathbf{k}_0, n)^{pr}$ *for every* $n \geq 2$.

Then: $\quad (\Omega_1, \mathbf{k}_1, \Gamma_1) \equiv_{\mathbf{k}_0 \cup \Gamma_0} (\Omega_2, \mathbf{k}_2, \Gamma_2) \iff \Gamma_1 \equiv_{\Gamma_0} \Gamma_2$.

4. DEFINABLE SETS IN $(\Omega, \mathbf{k}, \Gamma)$

A back-and-forth system. To define a back-and-forth system adequate for proving Theorem 1.2 is not so obvious, and we managed to do it only after much trial-and-error; see Lemmas 4.1, 4.2, 4.3. Throughout this subsection we fix a model $(\Omega_0, \mathbf{k}_0, \Gamma_0)$ of T such that $\mathbf{k}_0$ is infinite and $(\mathbf{k}_0, \Gamma_0)$ is a Mann pair.

Lemma 4.1. *Let* $(\Omega, \mathbf{k}, \Gamma)$ *be a model of* T *such that* $(\Omega_0, \mathbf{k}_0, \Gamma_0) \subseteq (\Omega, \mathbf{k}, \Gamma)$ *and* $\Gamma(\mathbf{k}, n)^{pr} = \Gamma_0(\mathbf{k}_0, n)^{pr}$ *for every* $n \geq 2$. *Then*

(1) $\mathbf{k}$ *and* $\mathbf{k}_0(\Gamma)$ *are linearly disjoint over* $\mathbf{k}_0$;
(2) *if* $\mathbf{k}|\mathbf{k}_0$ *is regular and* $\Gamma|\Gamma_0$ *is pure, then* $\mathbf{k}(\Gamma)|\mathbf{k}_0(\Gamma_0)$ *is regular.*

Proof. Let $\gamma_1, \ldots, \gamma_n \in \Gamma$ be linearly dependent over $\mathbf{k}$; to get (1) it is enough to show that then they are linearly dependent over $\mathbf{k}_0$. We can reduce to the case that $n \geq 2$ and $(\gamma_i)_{i \in I}$ is linearly independent over $\mathbf{k}$ for all nonempty proper subsets I of $\{1, \ldots, n\}$. Then by Corollary 3.5 we have

$$(\gamma_1, \ldots, \gamma_n) = \gamma \vec{\gamma}_0, \qquad \gamma \in \Gamma,\ \vec{\gamma}_0 \in \Gamma_0(\mathbf{k}_0, n)^{pr},$$

so $\gamma_1, \dots, \gamma_n$ are indeed linearly dependent over $\mathbf{k}_0$.

Suppose now that $\mathbf{k}|\mathbf{k}_0$ is regular and $\Gamma|\Gamma_0$ is pure. Then by (1) and Theorem 4.13 from [6] the extension $\mathbf{k}(\Gamma)|\mathbf{k}_0(\Gamma)$ is regular, and by Lemma 5.13 of [4] the extension $\mathbf{k}_0(\Gamma)|\mathbf{k}_0(\Gamma_0)$ is regular. Hence $\mathbf{k}(\Gamma)|\mathbf{k}_0(\Gamma_0)$ is regular, by Proposition 4.11(b) of Chapter VIII of [6]. □

Fix a cardinal $\kappa > |\Omega_0|$, and let $(\Omega, \mathbf{k}, \Gamma)$ be a κ-saturated model of T such that $(\Omega_0, \mathbf{k}_0, \Gamma_0) \subseteq (\Omega, \mathbf{k}, \Gamma)$ and $\Gamma(\mathbf{k}, n)^{pr} = \Gamma_0(\mathbf{k}_0, n)^{pr}$ for every $n \geq 2$. Define $\mathrm{Sub}(\Omega, \mathbf{k}, \Gamma)$ to be the collection of all models $(\Omega', \mathbf{k}', \Gamma')$ of T such that:

(i) $\Omega_0 \subseteq \Omega' \subseteq \Omega$ (as fields) and $|\Omega'| < \kappa$;
(ii) $\mathbf{k}_0 \subseteq \mathbf{k}' \subseteq \mathbf{k}$ (as fields) and the extension $\mathbf{k}|\mathbf{k}'$ is regular;
(iii) $\Gamma_0 \subseteq \Gamma' \subseteq \Gamma$ (as groups), and the extension $\Gamma|\Gamma'$ is pure;
(iv) $\mathbf{k}(\Gamma)$ and Ω' are free over $\mathbf{k}'(\Gamma')$.

So $(\Omega', \mathbf{k}', \Gamma') \in \mathrm{Sub}(\Omega, \mathbf{k}, \Gamma)$ yields a diagram of field inclusions:

$$\begin{tikzcd}
 & & \Omega \\
 & \mathbf{k}(\Gamma) & \Omega' \\
\mathbf{k} & \mathbf{k}'(\Gamma') & \\
\mathbf{k}' & &
\end{tikzcd}$$

Lemma 4.2. *Let* $(\Omega', \mathbf{k}', \Gamma') \in \mathrm{Sub}(\Omega, \mathbf{k}, \Gamma)$. *Then*

(1) $\mathbf{k}(\Gamma)$ *and* Ω' *are linearly disjoint over* $\mathbf{k}'(\Gamma')$*,*
(2) $\mathbf{k}$ *and* Ω' *are linearly disjoint over* $\mathbf{k}'$*,*
(3) $\mathbf{k}(\Gamma') \cap \Gamma = \Gamma'$*,*
(4) $(\Omega', \mathbf{k}', \Gamma')$ *is a substructure of* $(\Omega, \mathbf{k}, \Gamma)$*.*

Proof. From the linear disjointness of $\mathbf{k}$ and $\mathbf{k}_0(\Gamma)$ over $\mathbf{k}_0$—item (1) of Lemma 4.1—we obtain the linear disjointness of $\mathbf{k}$ and $\mathbf{k}'(\Gamma)$ over $\mathbf{k}'$. Hence $\mathbf{k}(\Gamma)|\mathbf{k}'(\Gamma)$ is regular. We then argue as in the proof of (2) in Lemma 4.1, with $\mathbf{k}'$ instead of $\mathbf{k}_0$, that $\mathbf{k}'(\Gamma)|\mathbf{k}'(\Gamma')$ is regular, and so $\mathbf{k}(\Gamma)|\mathbf{k}'(\Gamma')$ is regular. Then (1) follows from condition (iv) above and Theorem 4.12 on page 367 of [6]. This also gives (2).

For (3), let $\gamma \in \mathbf{k}(\Gamma') \cap \Gamma$, so $\gamma = \frac{a_1\alpha_1 + \cdots + a_m\alpha_m}{b_1\beta_1 + \cdots + b_n\beta_n}$, with $a_1, \dots, a_m, b_1, \dots, b_n$ from $\mathbf{k}$, and $\alpha_1, \dots, \alpha_m$, $\beta_1, \dots, \beta_n$ from Γ', and $b_1\beta_1 + \cdots + b_n\beta_n \neq 0$. Taking such a representation of γ with minimal $m+n$, we have $m, n \geq 1$, and using Corollary 3.5, this gives

$$(\alpha_1, \dots, \alpha_m, \beta_1\gamma, \dots, \beta_n\gamma) \in \Gamma\vec{\gamma}_0, \qquad \vec{\gamma}_0 \in \Gamma_0(\mathbf{k}_0, m+n).$$

In particular, $\frac{\beta_1 \gamma}{\alpha_1} \in \Gamma_0 \subseteq \Gamma'$, so $\gamma \in \Gamma'$.

As to (4), by (2) we have $\Omega' \cap \mathbf{k} = \mathbf{k}'$, so if $\gamma \in \Omega' \cap \Gamma$, then $\gamma \in \mathbf{k}'(\Gamma')$ by (1), and hence $\gamma \in \Gamma'$ by (3). $\square$

Next, for $i = 1, 2$, let $(\Omega_i, \mathbf{k}_i, \Gamma_i)$ be a κ-saturated model of T such that $(\Omega_0, \mathbf{k}_0, \Gamma_0) \subseteq (\Omega_i, \mathbf{k}_i, \Gamma_i)$, $[\Omega_i : \mathbf{k}_i] > 2$, and $\Gamma_i(\mathbf{k}_i, n)^{pr} = \Gamma_0(\mathbf{k}_0, n)^{pr}$ for every $n \geq 2$, and put $\mathrm{Sub}_i := \mathrm{Sub}(\Omega_i, \mathbf{k}_i, \Gamma_i)$.

Let $\mathcal{I}$ be the set of isomorphisms

$$\iota : (\Omega'_1, \mathbf{k}'_1, \Gamma'_1) \to (\Omega'_2, \mathbf{k}'_2, \Gamma'_2), \qquad (\Omega'_i, \mathbf{k}'_i, \Gamma'_i) \in \mathrm{Sub}_i \text{ for } i = 1, 2,$$

that are the identity on $\mathbf{k}_0$ and on Γ_0, such that $\iota|\mathbf{k}'_1$ is a partial elementary map from $\mathbf{k}_1$ to $\mathbf{k}_2$, and $\iota|\Gamma'_1$ is a partial elementary map from Γ_1 to Γ_2; we do *not* require that ι is the identity on Ω_0.

Lemma 4.3. *$\mathcal{I}$ is a (possibly empty) back-and-forth system.*

Proof. Let $\iota : (\Omega'_1, \mathbf{k}'_1, \Gamma'_1) \to (\Omega'_2, \mathbf{k}'_2, \Gamma'_2)$ be in $\mathcal{I}$, and $r \in \Omega_1 \setminus \Omega'_1$; by symmetry it is enough to show that then ι extends to an isomorphism in $\mathcal{I}$ that has r in its domain.

First, consider the case that $r \in \mathbf{k}_1$. Then we take a field $\mathbf{k}''_1 \preceq \mathbf{k}_1$ such that $\mathbf{k}'_1(r) \subseteq \mathbf{k}''_1$ and $|\mathbf{k}''_1| < \kappa$. Using saturation we extend $\iota|\mathbf{k}'_1$ to a field isomorphism $f : \mathbf{k}''_1 \to \mathbf{k}''_2 \preceq \mathbf{k}_2$ that is a partial elementary map between $\mathbf{k}_1$ and $\mathbf{k}_2$. It is clear that $\mathbf{k}_i|\mathbf{k}''_i$ is regular, and that $\mathbf{k}_i(\Gamma_i)$ and Ω''_i are free over $\mathbf{k}''_i(\Gamma'_i)$, where $\Omega''_i := (\Omega'_i \mathbf{k}''_i)^{\mathrm{ac}}$ and $i \in \{1, 2\}$. So $(\Omega''_i, \mathbf{k}''_i, \Gamma'_i) \in \mathrm{Sub}_i$ for $i = 1, 2$. Then by (2) of Lemma 4.2 we have a common extension of ι and f to an isomorphism

$$(\Omega''_1, \mathbf{k}''_1, \Gamma'_1) \to (\Omega''_2, \mathbf{k}''_2, \Gamma'_2)$$

in $\mathcal{I}$; it has r in its domain.

Next, assume that $r \in \Gamma_1$. Then we take a group $\Gamma''_1 \preceq \Gamma_1$ such that $r^{\mathbb{Z}}\Gamma'_1 \subseteq \Gamma''_1$ and $|\Gamma''_1| < \kappa$. Using saturation we extend $\iota|\Gamma'_1$ to a group isomorphism $g : \Gamma''_1 \to \Gamma''_2 \preceq \Gamma_2$ that is a partial elementary map between Γ_1 and Γ_2. It is clear that $\Gamma_i|\Gamma''_i$ is pure, and that $\mathbf{k}_i(\Gamma_i)$ and Ω''_i are free over $\mathbf{k}'_i(\Gamma''_i)$, where $\Omega''_i := \big(\Omega'_i(\Gamma''_i)\big)^{\mathrm{ac}}$ and $i \in \{1, 2\}$. So $(\Omega''_i, \mathbf{k}'_i, \Gamma''_i) \in \mathrm{Sub}_i$ for $i = 1, 2$. Also $\Gamma_i(\mathbf{k}_i, n) = \Gamma_0(\mathbf{k}_0, n)$, for $n > 1$ and $i = 1, 2$, and thus we have a field isomorphism $h : \mathbf{k}'_1(\Gamma''_1) \to \mathbf{k}'_2(\Gamma''_2)$ that extends $\iota|\mathbf{k}'_1$ and g. Then by (1) of Lemma 4.2 this gives a common extension of ι and h to an isomorphism

$$(\Omega''_1, \mathbf{k}'_1, \Gamma''_1) \to (\Omega''_2, \mathbf{k}'_2, \Gamma''_2)$$

in $\mathcal{I}$; it has r in its domain.

If $r \in \Omega'_1(\mathbf{k}_1 \cup \Gamma_1)^{\mathrm{ac}}$, then we can take a finite number of steps of the two types above to extend ι to an element of $\mathcal{I}$ with r in its domain.

Finally, suppose that $r \notin \Omega'_1(\mathbf{k}_1 \cup \Gamma_1)^{\mathrm{ac}}$. By saturation and the smallness assumption $[\Omega_2 : \mathbf{k}_2] > 2$ we can take $s \in \Omega_2$ with $s \notin \Omega'_2(\mathbf{k}_2 \cup \Gamma_2)^{\mathrm{ac}}$. With

$$\Omega''_1 := \Omega'_1(r)^{\mathrm{ac}}, \qquad \Omega''_2 := \Omega'_2(s)^{\mathrm{ac}},$$

it is clear that $(\Omega_i'', \mathbf{k}_i', \Gamma_i') \in \mathrm{Sub}_i$ for $i = 1, 2$. We can extend ι to a field isomorphism $\Omega_1'' \to \Omega_2''$ that sends r to s, and this gives an isomorphism

$$(\Omega_1'', \mathbf{k}_1', \Gamma_2') \to (\Omega_2'', \mathbf{k}_2', \Gamma_2')$$

in $\mathcal{I}$ with r in its domain. $\square$

Corollary 4.4. *Suppose that* $(\Omega_0, \mathbf{k}_0, \Gamma_0) \subseteq (\Omega, \mathbf{k}, \Gamma) \models T$ *and* $[\Omega_0 : \mathbf{k}_0] > 2$. *Then* $(\Omega_0, \mathbf{k}_0, \Gamma_0) \preceq (\Omega, \mathbf{k}, \Gamma)$ *if and only if* $(1) - (4)$ *below are satisfied:*

1. $[\Omega : \mathbf{k}] > 2$*;*
2. $\Gamma(\mathbf{k}, n)^{pr} = \Gamma_0(\mathbf{k}_0, n)^{pr}$ *for every* $n \geq 2$*;*
3. $\mathbf{k}_0 \preceq \mathbf{k}$ *and* $\Gamma_0 \preceq \Gamma$*;*
4. $\mathbf{k}(\Gamma)$ *and* Ω_0 *are free over* $\mathbf{k}_0(\Gamma_0)$.

Proof. It is easy to check that if $(\Omega_0, \mathbf{k}_0, \Gamma_0) \preceq (\Omega, \mathbf{k}, \Gamma)$, then (1)–(4) hold. For the converse, assume (1)-(4). By passing to an elementary extension of $(\Omega, \mathbf{k}, \Gamma)$ we arrange that $(\Omega, \mathbf{k}, \Gamma)$ is κ-saturated. Put

$$(\Omega_1, \mathbf{k}_1, \Gamma_1) := (\Omega, \mathbf{k}, \Gamma)$$

and let $(\Omega_2, \mathbf{k}_2, \Gamma_2)$ be a κ-saturated elementary extension of $(\Omega_0, \mathbf{k}_0, \Gamma_0)$. Then $(\Omega_0, \mathbf{k}_0, \Gamma_0) \in \mathrm{Sub}(\Omega_i, \mathbf{k}_i, \Gamma_i)$ for $i = 1, 2$. Let $\mathcal{I}$ be the back-and-forth system considered in the previous lemma. Then the identity map on $(\Omega_0, \mathbf{k}_0, \Gamma_0)$ belongs to $\mathcal{I}$, and so $(\Omega_1, \mathbf{k}_1, \Gamma_1)$ and $(\Omega_2, \mathbf{k}_2, \Gamma_2)$ are elementarily equivalent over Ω_0. Thus $(\Omega_0, \mathbf{k}_0, \Gamma_0) \preceq (\Omega, \mathbf{k}, \Gamma)$. $\square$

Definable Sets. We now specify the two unary relation symbols of $\mathcal{L}$ to be U and V, to be interpreted in a model of T as the underlying set of the distinguished subfield and of the distinguished multiplicative group, respectively. Let a model $(\Omega, \mathbf{k}, \Gamma)$ of T be given and let $x = (x_1, \ldots, x_m)$ be a tuple of distinct variables. Call a subset of Ω^m *special* if it is defined in $(\Omega, \mathbf{k}, \Gamma)$ by a special formula in $x = (x_1, \ldots, x_m)$, that is, a formula

$$\exists y \; \exists z (U(y) \wedge V(z) \wedge \phi(x, y, z)),$$

where $x_1, \ldots, x_m, y_1, \ldots, y_s, z_1, \ldots, z_t$ are distinct variables, $y = (y_1, \ldots, y_s)$, $z = (z_1, \ldots, z_t)$ and $\phi(x, y, z)$ is a quantifier-free formula in the language of rings augmented by names for the elements of Ω, and where $U(y)$ and $V(z)$ abbreviate $U(y_1) \wedge \cdots \wedge U(y_s)$ and $V(z_1) \wedge \cdots \wedge V(z_t)$, respectively.

Now we are ready to prove Theorem 1.2, which we first reformulate using the above terminology.

Theorem 4.5. *Let* $(\Omega, \mathbf{k}, \Gamma)$ *be a model of* T *such that* $\mathbf{k}$ *is an algebraically closed field,* $\mathbf{k} \neq \Omega$*,* $(\mathbf{k}, \Gamma)$ *is a Mann pair, and* $\Gamma / \Gamma^{[p]}$ *is finite for each* p. *Then the subsets of* Ω^m *definable in* $(\Omega, \mathbf{k}, \Gamma)$ *are exactly the boolean combinations in* Ω^m *of special subsets of* Ω^m.

Proof. We take $\kappa := \aleph_1$, and may assume that $(\Omega, \mathbf{k}, \Gamma)$ is κ-saturated. Let $(\Omega_0', \mathbf{k}_0, \Gamma_0)$ be a countable elementary substructure of $(\Omega, \mathbf{k}, \Gamma)$, and let Ω_0 be the algebraic closure of $\mathbf{k}_0(\Gamma_0)$ in Ω_0'. Then $(\Omega_0, \mathbf{k}_0, \Gamma_0) \subseteq (\Omega, \mathbf{k}, \Gamma)$,

$(\mathbf{k}_0, \Gamma_0)$ is a Mann pair, and $\Gamma(\mathbf{k}, n)^{pr} = \Gamma_0(\mathbf{k}_0, n)^{pr}$ for every $n \geq 1$. Let $\vec{r} = (r_1, \ldots, r_m)$ and $\vec{s} = (s_1, \ldots, s_m)$ be two tuples from Ω^m that satisfy the same special formulas in x using only names for elements of $A := \mathbf{k}_0 \cup \Gamma_0$; it suffices to show that then they realize the same type in $(\Omega, \mathbf{k}, \Gamma)$ over A.

For $i = 1, 2$, put $(\Omega_i, \mathbf{k}_i, \Gamma_i) := (\Omega, \mathbf{k}, \Gamma)$. Hence all the structures in $\mathrm{Sub}(\Omega_i, \mathbf{k}_i, \Gamma_i)$ are countable. Set $\mathcal{I}$ to be the back-and-forth system of Lemma 4.3; it is enough to construct an isomorphism in this system that takes $\vec{r}$ to $\vec{s}$.

Let d be the transcendence degree of $\mathbf{k}(\Gamma)(\vec{r})$ over $\mathbf{k}(\Gamma)$. We can assume that $r_1, \ldots, r_d$ are algebraically independent over $\mathbf{k}(\Gamma)$. As in the proof of Theorem 3.8 in [4] it follows that $s_1, \ldots, s_d$ are algebraically independent over $\mathbf{k}(\Gamma)$, and $s_{d+1}, \ldots, s_m$ are algebraic over $\mathbf{k}(\Gamma)(s_1, \ldots, s_d)$.

Take some countable $(\Omega', \mathbf{k}', \Gamma') \preceq (\Omega, \mathbf{k}, \Gamma)$ such that $\Omega' \supseteq \Omega_0$ and $\mathbf{k}'(\Gamma')(\vec{r})$ has transcendence degree d over $\mathbf{k}'(\Gamma')$. In particular, $(\Omega', \mathbf{k}', \Gamma') \in \mathrm{Sub}(\Omega, \mathbf{k}, \Gamma)$. Let $a = (a_0, a_1, a_2, \ldots)$ be an enumeration of $\mathbf{k}'$, let $g = (g_0, g_1, g_2, \ldots)$ be an enumeration of Γ', and let $y_0, y_1, y_2, \ldots, z_0, z_1, z_2, \ldots$ be distinct variables, also distinct from $x_1, \ldots, x_m$, and put

$$y = (y_0, y_1, y_2, \ldots), \qquad z = (z_0, z_1, z_2, \ldots,).$$

Suppose $\psi_1(y), \ldots, \psi_k(y)$ are quantifier-free formulas in the language of rings augmented by names for the elements of $\mathbf{k}_0$, and $\theta_1(z), \ldots, \theta_k(z)$ are quantifier-free formulas in the language of groups augmented by names for the elements of Γ_0, and $\phi_1(x, y, z), \ldots, \phi_k(x, y, z)$ are quantifier-free formulas in the language of rings augmented by names for the elements of A, such that $\mathbf{k} \models \psi_j(a)$, $\Gamma \models \theta_j(g)$ and $(\Omega, \mathbf{k}, \Gamma) \models \phi_j(\vec{r}, a, g)$ for $j = 1, \ldots, k$. Then

$$(\Omega, \mathbf{k}, \Gamma) \models \exists y \exists z (U(y) \wedge V(z) \wedge \psi(y) \wedge \theta(z) \wedge \phi(\vec{r}, y, z)), \quad \text{where}$$

$$\psi(y) := \bigwedge_j \psi_j(y), \quad \theta(z) := \bigwedge_j \theta_j(z), \quad \varphi(x, y, z) := \bigwedge_j \varphi_j(x, y, z).$$

The assumption on $\vec{r}$ and $\vec{s}$ then gives

$$(\Omega, \mathbf{k}, \Gamma) \models \exists y \exists z (U(y) \wedge V(z) \wedge \psi(y) \wedge \theta(z) \wedge \phi(\vec{s}, y, z)).$$

Hence we have a partial y, z-type over $A\vec{s}$ in $(\Omega, \mathbf{k}, \Gamma)$ consisting of the formulas $U(y_i)$ and $V(z_i)$ for $i = 0, 1, 2, \ldots$, the quantifier-free formulas $\psi(y)$ in the language of rings augmented by names for the elements of $\mathbf{k}_0$ such that $\mathbf{k} \models \psi(a)$, the quantifier-free formulas $\theta(z)$ in the language of groups augmented by names for the elements of Γ_0 such that $\Gamma \models \theta(g)$, and the formulas $\phi(\vec{s}, y, z)$ such that $\phi(x, y, z)$ is a quantifier-free formula in the language of rings augmented by names for the elements of A and $(\Omega, \mathbf{k}, \Gamma) \models \phi(\vec{r}, a, g)$. Let b, h with $b = (b_0, b_1, b_2, \ldots) \in \mathbf{k}^{\mathbb{N}}$ and $h = (h_0, h_1, h_2, \ldots) \in \Gamma^{\mathbb{N}}$ realize this y, z-type in $(\Omega, \mathbf{k}, \Gamma)$. Then $\{b_0, b_1, b_2, \ldots\}$ is the underlying set of a field $\mathbf{k}'' \preceq \mathbf{k}$ and we have a field isomorphism

$$\iota^{\mathrm{f}} : \mathbf{k}' \to \mathbf{k}'', \qquad \iota^{\mathrm{f}}(a_n) = b_n \text{ for all } n.$$

Likewise, $\{h_0, h_1, h_2, \dots\}$ is the underlying set of a group $\Gamma'' \preceq \Gamma$ and we have a group isomorphism

$$\iota^{\mathrm{g}} : \Gamma' \to \Gamma'', \qquad \iota^{\mathrm{g}}(g_n) = h_n \text{ for all } n.$$

Note that ι^{f} is a partial elementary map from $\mathbf{k}$ to itself and is the identity on $\mathbf{k}_0$. Likewise, ι^{g} is a partial elementary map from Γ to itself and is the identity on Γ_0. Moreover, ι^{f} and ι^{g} have a common extension to a field isomorphism

$$\iota : \mathbf{k}'(\Gamma')(\vec{r}) \cong \mathbf{k}''(\Gamma'')(\vec{s})$$

sending $\vec{r}$ to $\vec{s}$. Put $\Omega_1' := \mathbf{k}'(\Gamma')(\vec{r})^{\mathrm{ac}}$ and $\Omega_2' := \mathbf{k}''(\Gamma'')(\vec{s})^{\mathrm{ac}}$. Then $(\Omega_1', \mathbf{k}', \Gamma'), (\Omega_2', \mathbf{k}'', \Gamma'') \in \mathrm{Sub}(\Omega, \mathbf{k}, \Gamma)$, and we have an isomorphism

$$(\Omega_1', \mathbf{k}', \Gamma') \cong (\Omega_2', \mathbf{k}'', \Gamma'')$$

that extends ι. It carries $\vec{r}$ to $\vec{s}$ and belongs to $\mathcal{I}$. $\square$

Note that if Γ is divisible or of finite rank, then the condition in the theorem that $\Gamma/\Gamma^{[p]}$ is finite for each p is satisfied.

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