HELLY'S THEOREM, DOLICH'S THEOREM AND SO ON

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1. Introduction

Throughout $\mathcal{R} = (R, +, \cdot, \dots)$ is an o-minimal field; even though most of the results hold for any o-minimal structure with definable Skolem functions.

We start by stating the following celebrated theorem, which classically known for $R = \mathbb{R}$, but the proof goes through for any R. We do not give a proof here.

Theorem 1.1. *(Helly) Let $A_1, \dots, A_k$ be a collection of convex subsets of R^n with the property that any $n+1$ of them have a nonzero intersection. Then the intersection $A_1 \cap \dots \cap A_k$ is nonempty.*

(We refer to the property assumed for the sets A_i in this theorem as $n+1$-*intersection property*.)

Some heuristic: this theorem is something like a *quantitative version of compactness theorem*. If there is a contradicton, then it has to happen before the $n+1$-st step. Also note that it is not correct without the word 'convex' (Union of two intervals).

We prove the following easy consequence of Helly's theorem before we go further.

Corollary 1.2. *Let $\mathfrak{C}$ be a family of convex subsets of R^n and $p_1, \dots, p_k$ such that for every $C_1, \dots, C_{n+1} \in \mathfrak{C}$ there is $i \in \{1, \dots, k\}$ such that $p_i \in \bigcap_{j=1}^{n+1} C_j$. Then $\bigcap \mathfrak{C} \neq \emptyset$.*

Proof. Consider the (finite) family $\mathfrak{P}$ consisting of the convex hulls of $C \cap \{p_1, \dots, p_k\}$ with $C \in \mathfrak{C}$. By assumption $\mathfrak{P}$ has the $n+1$-intersection property. Therefore by Theorem 1.1, $\bigcap \mathfrak{P} \neq \emptyset$, and thus $\bigcap \mathfrak{C} \neq \emptyset$ since $\bigcap \mathfrak{P} \subseteq \bigcap \mathfrak{C}$. $\square$

Today we obtain a proof of the following *infinite* version of Theorem 1.1. When doing so we put together some material from different sources, but we mostly follow Section 3.2 of [1].

Theorem 1.3. *Let $\mathfrak{C}$ be a definable collection of convex, closed and bounded subsets of R^n with the n-intersection property. Then $\bigcap \mathfrak{C} \neq \emptyset$.*

Our proof depends on a theorem of Dolich/Peterzil-Pillay/Starchenko that can be stated only after introducing some model theoretic concepts. We do this in the next section.

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2. Forking and Dolich's Theorem

In the beginning of this section T is an arbitrary theory and $\mathbb{M}$ is a monster model of it. The following notation will be convenient: for a tuple of variables x, we let $\mathbb{M}_x$ denote $\mathbb{M}^{|x|}$.

Definition 2.1. Let $A \subseteq \mathbb{M}$, $\phi(x, y)$ a formula and $b \in \mathbb{M}_y$.

(1) The formula $\phi(x, b)$ *divides over* A if there is an indiscernible sequence $(b_i)_{i\in\mathbb{N}}$ over A such that $b_0 = b$ and the set $\{\phi(\mathbb{M}, b_i) : i \in \mathbb{N}\}$ is k-inconsistent for some k.

(2) The formula $\phi(x, b)$ *forks over* A if it implies (in T) a disjunction of formulas, each dividing over A.

(3) Let $B \supset A$ and $p \in S_y(B)$. Then the type p *forks over* A if it contains a formula that forks over A.

Now let T be an o-minimal theory extending RCF. The following is a simpler version of Dolich's theorem from [2] as it appears in an unpublished note by Starchenko ([3]).

Theorem 2.2. *Let $M \preceq \mathbb{M}$, $\phi(x, y)$ a formula and $b \in \mathbb{M}_y$. Then $\phi(x, b)$ does not fork over M if and only if $\phi(\mathbb{M}, b)$ has a point in $M\langle\delta\rangle$ for every (some) $\delta > M\langle b\rangle$.*

We have the following consequence when the set $\phi(\mathbb{M}, b)$ is closed and bounded.

Corollary 2.3. *Suppose, in addition to the assumptions of the theorem above, that $\phi(\mathbb{M}, b)$ is closed and bounded. Then $\phi(x, b)$ does not fork over M if and only if $\phi(\mathbb{M}, b)$ has a point in M.*

The useful (for us) consequence of this corollary is the following.

Proposition 2.4. *Let M be a small model of T and let $\mathfrak{F}$ be a definable family of closed and bounded subsets of R^n. If $\mathfrak{F}$ has the k-intersection property for some sufficiently large k, then there is a finite set $\mathcal{P} \subseteq R^n$ such that for all $F \in \mathfrak{F}$ we have $F \cap \mathcal{P} \neq \emptyset$.*

The problem is the meaning of *sufficiently large* in this statement. In order to understand it, I don't see a way of avoiding VC-density, which we study in the next section.

3. VC-stuff

Let X be a set and $\mathcal{S}$ a collection of subsets of it. We define the following function.

$$\pi_{\mathcal{S}} : \mathbb{N} \to \mathbb{N} : n \mapsto \max\{|\mathcal{S} \cap A| : A \in \binom{X}{n}\},$$

where $\mathcal{S} \cap A := \{S \cap A : S \in \mathcal{S}\}$ and $\binom{X}{n}$ is the collection of subsets of X with n elements.

Clearly $0 \leq \pi_{\mathcal{S}}(n) \leq 2^n$. If there is $n \in \mathbb{N}$ such that for all $A \in \binom{X}{n}$ with $\mathcal{S} \cap A \neq \mathcal{P}(A)$, then we say that $\mathcal{S}$ is a *VC-class* and the largest such n is called the *VC-dimension of* $\mathcal{S}$; denoted by $\mathrm{VC}(\mathcal{S})$. Here is a very surprising (at least it was for me) result.

Theorem 3.1. *(Sauer) Either $\pi_{\mathcal{S}}(n) = 2^n$ for all n or there is $d > 0$ such that $\pi_{\mathcal{S}}(n) = O(n^d)$. Indeed, this d is exactly* $\mathrm{VC}(\mathcal{S})$.

The notion of *VC-density* is a little bit more refined version of the VC-dimension; namely it is the infimum of $r \in \mathbb{R}^{>0}$ such that $\frac{\pi_{\mathcal{S}}(n)}{n^r}$ is bounded. It is denoted by $\mathrm{vc}(\mathcal{S})$. By the theorem above, we have $\mathrm{vc}(\mathcal{S}) \leq \mathrm{VC}(\mathcal{S})$.

Next we define the dual notions. For X and $\mathcal{S}$ above and $A_1, \ldots, A_n \subseteq X$ define $\mathcal{S}(A_1, \ldots, A_n)$ to be the collection of subsets of X of the form

$$\bigcap_{i \in I} A_i \cap \bigcap_{j \notin I} X \setminus A_j,$$

for various $I \subseteq n$. This allows us to define

$$\pi^*_{\mathcal{S}}(n) := \max\{|\mathcal{S}(A_1, \ldots, A_n)| : A_1 \ldots, A_n \subseteq X\}.$$

We also define $\mathrm{VC}^*(\mathcal{S})$ and $\mathrm{vc}^*(\mathcal{S})$ as before, $\pi_{\mathcal{S}}$ replaced by $\pi^*_{\mathcal{S}}$. Note that, even though there is no obvious numerical relation between $\mathrm{vc}(\mathcal{S})$ and $\mathrm{vc}^*(\mathcal{S})$, they are either both infinite or both finite.

We use these notions in the case $\mathcal{S} = \{\phi(\mathbb{M}, b) : b \in \mathbb{M}_y\}$, where $\mathbb{M}$ is a monster model of a first order theory T and $\phi(x, y)$ is a formula in its language. In this case the relation between $\mathrm{vc}(\mathcal{S})$ and $\mathrm{vc}^*(\mathcal{S})$ is clearer. Let $\mathcal{S}^*$ be

$$\{\phi(a, \mathbb{M}) : a \in \mathbb{M}_x\}.$$

Then $\pi^*_{\mathcal{S}} = \pi_{\mathcal{S}^*}$ and vice-versa. From now one we write $\mathrm{vc}(\phi)$ rather than $\mathrm{vc}(\mathcal{S})$. The next lemma is crucial to get Proposition 2.4 from Corollary 2.3.

Lemma 3.2. *Let $d = \lfloor \mathrm{vc}^*(\phi) \rfloor + 1$ and $(b_i)_{i \in \mathbb{N}}$ an indiscernible sequence over $\emptyset$ from $\mathbb{M}_y$. Also suppose that set $\mathcal{T} := \{\phi(\mathbb{M}, b_i) : i \in \mathbb{N}\}$ has the d-intersection property. Then $\bigcap T \neq \emptyset$.*

Proof. Assume that $\bigcap T = \emptyset$. Then by saturation, there is k such that $\phi(\mathbb{M}, b_0) \cap \cdots \cap \phi(\mathbb{M}, b_k) = \emptyset$. Let D be the minimum such k. So $D \geq d$.

So by indiscernibility, for all $I_0 \in \binom{\mathbb{N}}{D}$ the set $\{\phi(\mathbb{M}, b_i) : i \in I_0\}$ is consistent and for all $I_1 \in \binom{\mathbb{N}}{D+1}$ the set $\{\phi(\mathbb{M}, b_i) : i \in I_1\}$ is inconsistent.

Let $t > D$. For $I \in \binom{t}{D}$ consider the set

$$X_I \bigcap_{i \in I} \phi(\mathbb{M}, b_i) \cap \bigcap_{j \notin I} \phi(\mathbb{M}, b_j).$$

For each such I, this set is nonempty and for $I \neq J$ we have $X_I \neq X_J$. But this means that $\pi^*_{\phi}(t)$ is strictly greater than t^D and hence greater than t^d, contradicting the definition of d. □

4. Proof of Theorem 1.3

We first give a proof of Proposition 2.4 from Corollary 2.3: Let $\mathfrak{F}$ be defined by $\phi(x, y)$ and assume that the conclusion of the proposition does not hold. Then by compactness there is a big model $\mathbb{M}$ of $\mathrm{Th}(\mathfrak{R})$ and an element $b \in \mathbb{M}_y$ such that $\phi(\mathbb{M}, b) \cap R_x = \emptyset$. So by Corollary 2.3, the formula $\phi(x, b)$ forks over R. But by Lemma 3.2, once $\mathfrak{F}$ has the d-intersection property, it has D-intersection property for all $D \geq d$, which is a contradiction.

In order to finish the proof of Theorem 1.3, let $\mathfrak{D}$ be the collection of subsets of R^n consisting of $C_1 \cap \cdots \cap C_{n+1}$ where $C_i \in \mathfrak{C}$. Now by finite Helly Theorem, $\mathfrak{D}$ has the $n+1$-intersention property. Hence by Proposition 2.4, there is a finite subset

$\mathcal{P}$ of R^n such that for each $D \in \mathfrak{D}$, we have $D \cap \mathcal{P} \neq \emptyset$. But this finishes the proof using Lemma 1.2.

References

[1] Matthias Aschenbrenner and Andreas Fischer. Definable versions of theorems by Kirszbraun and Helly. *Proc. Lond. Math. Soc. (3)*, 102(3):468–502, 2011.

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[3] Sergei Starchenko. A note on dolich's paper. Available at `http://www.logique.jussieu.fr/modnet/Publications/Preprint server`, 2008.

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