DEPENDENT PAIRS

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ABSTRACT. We prove that certain pairs of ordered structures are dependent. Among these structures are dense and tame pairs of o-minimal structures and further the real field with a multiplicative subgroup with the Mann property.

1. INTRODUCTION

The *dependence property* is first introduced by S. Shelah in [S71]. The basics of this property can be found in [P00]. Recently there has been some work on the dependence property and its variants, mostly in the setting of abstract model theory. Also in connection with the solution of *o-minimal conjectures* of A.Pillay, groups definable in dependent theories are considered in [HPP06].

One reason for the interest in the dependent theories is that they generalize both the stable and o-minimal theories. Actually a theory is stable if and only if it does not have the strict order property and it is dependent.

In the present paper we construct examples of dependent theories appearing in a natural way. We show that the theories of dense pairs of o-minimal structures (see [vdD98]) and tame pairs of o-minimal structures (see [vdDL95]) are dependent. Actually our techniques apply to more general pairs. For instance the theory of a real closed field R expanded by a dense multiplicative subgroup of $R^{>0}$ with the Mann property is dependent whenever the group has the property that for every prime p the subgroup of p^{th} powers has finite index. Moreover if we augment this last structure by adding certain power functions we still get a dependent theory. Finally we show that a real closed field expanded by a cyclic multiplicative subgroup generated by a positive element is dependent.

Here is a precise definition of the property under question.

Definition 1.1. Let T be a complete theory in the language $\mathcal{L}$, and let $\mathbb{M}$ be a monster model of T.

(1) Let $\varphi(\vec{x}, \vec{y})$ be an $\mathcal{L}$-formula, where $\vec{x} = (x_1, \ldots, x_m)$ and $\vec{y} = (y_1, \ldots, y_p)$ are tuples of variables. We say that $\varphi(\vec{x}, \vec{y})$ is *dependent* (in T) if for every indiscernible sequence $(\vec{a_i})_{i\in\mathbb{N}}$ of tuples of length m and every $\vec{b} \in \mathbb{M}^p$ there is $N \in \mathbb{N}$ such that either $\mathbb{M} \models \varphi(\vec{a_i}, \vec{b})$ for every $i > N$ or $\mathbb{M} \models \neg\varphi(\vec{a_i}, \vec{b})$ for every $i > N$.

(2) The theory T is *dependent* if every $\mathcal{L}$-formula is dependent in T

One of the key facts we use from [S71] is that to check whether a theory is dependent, it is enough to check all formulas of the form $\varphi(x, \vec{y})$ are dependent.

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Throughout $\mathcal{A} = (A, <, \dots)$ will be an o-minimal expansion of a densely ordered abelian group in the language $\mathcal{L}$ and B a subset of A. We consider the structure $(\mathcal{A}, B)$ in the language $\mathcal{L}(U) := \mathcal{L} \cup \{U\}$, where U is a unary relation symbol not in $\mathcal{L}$. We basically have two cases according to B being dense or discrete in its convex hull in A. We handle these cases separately.

Our examples in the first case have the common property that their open core is o-minimal. Hence we are obliged to give a precise definition of the open core of an ordered structure.

Definition 1.2. Let $\mathcal{R} = (R, <, \dots)$ be an expansion of a dense linear order without endpoints. Then the *open core*, $\mathcal{R}^\circ$ is the structure $\big(\mathcal{R}, (U)\big)$ where U runs through the collection of open definable subsets of R^n for various $n \in \mathbb{N}$.

Let B be dense in A and define T_B to be the theory of $(\mathcal{A}, B)$. Our main theorem in this case is is as follows.

Theorem 1.3. *Suppose that for every model $(\mathcal{M}, N)$ of T_B the following hold:*

(i) *every subset of N^n definable in $(\mathcal{M}, N)$ is a boolean combination of sets of the form $S \cap K$, where $S \subseteq M^n$ is definable in $\mathcal{M}$ and $K \subseteq M^n$ is $\emptyset$-definable in $(\mathcal{M}, N)$,*

(ii) *every subset of M^m definable in $(\mathcal{M}, N)$ is a boolean combination of subsets of M^m defined by*
$$\exists y_1 \cdots \exists y_n \; U(y_1) \wedge \cdots \wedge U(y_n) \wedge \varphi(x_1, \dots, x_m, y_1, \dots, y_n),$$
where φ is a quantifier-free $\mathcal{L}$-formula (possibly with parameters from M),

(iii) *$(\mathcal{M}, N)$ has o-minimal open core.*

Then T_B is dependent.

We prove this theorem in the next section after some preparation.

For the second case assume that $\mathcal{A}$ expands a real closed field. We handle this case in a different language: we extend $\mathcal{L}$ by a unary function symbol ϕ to get $\mathcal{L}(\phi)$.

Theorem 1.4. *Let $T(\phi)$ be an $\mathcal{L}(\phi)$-theory extending the theory of $\mathcal{A}$ and $\mathbb{M}$ a monster model of it. Suppose that*

(i) *the theory $T(\phi)$ has quantifier elimination,*

(ii) *for every $(\mathcal{M}, \phi) \models T(\phi)$ and $\mathcal{N} \preceq \mathcal{M}$ with $\phi(N) \subseteq N$ and every $m_1, \dots, m_n$ from M*
$$\dim \phi\big(N\langle m_1, \dots, m_n\rangle\big) - \dim \phi(N) \le n,$$
where $\dim$ is the dimesion notion from the pregeometry given by the definable closure with respect to the language $\mathcal{L}$, and $N\langle m_1, \dots, m_n\rangle$ is the definable closure of N and $m_1, \dots, m_n$ in this pregeometry,

(iii) *for all functions $f : \mathbb{M}^{n+k} \to \mathbb{M}$ and $g : \mathbb{M}^{j+l} \to \mathbb{M}$, both $\emptyset$-definable in the $\mathcal{L}$-reduct of $\mathbb{M}$, for every indiscernible sequence $(\vec{a}_i)_{i \in \mathbb{N}}$ with $a_{i,1}, \dots, a_{i,j} \in \phi(\mathbb{M})$, and tuples $\vec{b_1} \in \mathbb{M}^k$ and $\vec{b_2} \in (\phi(\mathbb{M}))^l$, the set*
$$\big\{ i \in \mathbb{N} \;:\; \mathbb{M} \models \phi(f(\vec{a}_i, \vec{b_1})) = g(a_{i,1}, \dots, a_{i,j}, \vec{b_2}) \big\}$$
is finite or cofinite.

Then $T(\phi)$ is dependent.

We have learned that A. Berenstein, A. Dolich and A. Onshuus have been working on similar topics. However, through a personal communication of the first author with A. Berenstein we are convinced that there are enough difference between the two projects. Also G. Boxall have shown some parallel results in his thesis ([B09]). The pairs he considers are closer to the ones considered by the authors mentioned above, and his techniques are quite different than ours.

Notations, conventions. Throughout m, n range over $\mathbb{N} := \{0, 1, 2, \dots\}$ the set of natural numbers, and *'definable'* means 'definable with parameters'.
We name model theoretic structures with capital letters in Calligraphic font, and the underlying set of these structures with the same capital letter in the normal font, with the exception that a monster model of a theory is denoted by a capital letter in blackboard bold font, although we reserve $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ for their standard use. For instance $\mathcal{R} = (R, \dots)$ will denote an arbitrary model theoretic structure with R as the underlying set, whereas $\mathbb{M}$ will denote a monster model of a theory. We use letters x, y, z for variables and letters a, b, c for elements from the underlying set of a structure. We distinguish tuples of variables from a single variable by using vector notation, likewise for tuples of elements. For example x is a single variable and $\vec{x}$ is a tuple of variables.

1.1. **Two combinatorial facts.** In the sequel we need two combinatorial results for indiscernible sequences. The first one is a corollary of Erdös-Rado Theorem which was observed by Shelah in [S80], and the second one is folklore, but we include a proof for completeness.

Theorem 1.5. *Let T be a theory and $\mathbb{M}$ a monster model of it. Also let $\vec{b}$ be a tuple of elements from $\mathbb{M}$ and $(\vec{a_i})_{i<\kappa}$ any sequence of tuples from $\mathbb{M}$, where $\kappa := \beth_{(2^{|T|+|\vec{b}|})^+}$. Then there is an indiscernible sequence $\left(\vec{a'_i}\right)_{i\in\mathbb{N}}$ such that given $k \in \mathbb{N}$ there are $i_1 < \dots < i_k$ with*

$$\mathbb{M} \models \varphi(\vec{a'_1}, \dots, \vec{a'_k}, \vec{b}) \text{ iff } \mathbb{M} \models \varphi(\vec{a_{i_1}}, \dots, \vec{a_{i_k}}, \vec{b})$$

for every parameter-free formula φ.

In the following, $\mathbb{M}$ is a monster model of T_B as in the setting of Theorem 1.3.

Lemma 1.6. Let $(a_n)_{n\in\mathbb{N}}$ be an indiscernible sequence such that for every $n \in \mathbb{N}$ $a_n \in \mathrm{dcl}_{\mathcal{L}}\left(U(\mathbb{M}) \cup \{a_1, \dots, a_N\}\right)$ for some $N \in \mathbb{N}$. Then there exist a function $f : \mathbb{M}^m \to \mathbb{M}$ definable in $\mathbb{M}$ over $a_1, \dots, a_N$ and an indiscernible sequence $(\vec{g}_n)_{n>N}$ such that for every $n > N$ we have $\vec{g}_n \in U(\mathbb{M})^m$ and $f(\vec{g}_n) = a_n$.

Proof: By the indiscernibility of $(a_n)_{n\in\mathbb{N}}$ there is a function $f : \mathbb{M}^m \to \mathbb{M}$ definable in $\mathbb{M}$ over $a_1, \dots, a_N$ such that for every $n \in \mathbb{N}$ there is $\vec{h}_n \in U(\mathbb{M})^m$ with

$$f(\vec{h}_n) = a_n. \tag{1.1}$$

We want to show that there is an indiscernible sequence $(\vec{g_n})_{n>N}$ such that (1.1) holds for every $n > N$ with $\vec{g}_n$ in the place of $\vec{h}_n$.

Put $\vec{y} = (\vec{y}_n)_{n>N}$. Let $\Sigma_1(\vec{y})$ be the collection of formulas

$$\psi(\vec{y}_{i_1}, \dots, \vec{y}_{i_p}) \leftrightarrow \psi(\vec{y}_{j_1}, \dots, \vec{y}_{j_p}),$$

where $\psi(\vec{z}_1, \dots, \vec{z}_p)$ is an $\mathcal{L}(U)$-formula and $i_1 < \dots < i_p$ and $j_1 < \dots < j_p$ are natural numbers larger N, and let $\Sigma_2(\vec{y})$ be the collection of formulas

$$f(a_1, \dots, a_N, \vec{y}_n) = a_n \wedge U(\vec{y}_n)$$

for $n > N$.

Now the desired indiscernible sequence is a realization of $\Sigma := \Sigma_1 \cup \Sigma_2$ and by saturation, it is just left to show that Σ is finitely satisfiable in $\mathbb{M}$. So let Δ_1 and Δ_2 be finite subsets of Σ_1 and Σ_2 respectively. Let I be the finite subset of $\mathbb{N}^{>N} := \{N, N+1, N+2, \dots\}$ such that if $\vec{y_n}$ occurs in $\Delta_1 \cup \Delta_2$, then $n \in I$. Further let Λ be the finite set of $\mathcal{L}(U)$-formulas such that for every $\psi \in \Lambda$ there are $i_1 < \dots < i_p$ and $j_1 < \dots < j_p$ natural numbers larger N and

$$\psi(y_{i_1}, ..., y_{i_p}) \leftrightarrow \psi(y_{j_1}, ..., y_{j_p}) \in \Delta_1.$$

For $i_1 < \dots < i_p$ put

$$X(i_1, \dots, i_p) := \{\psi \in \Lambda \ : \ \mathbb{M} \models \psi(\vec{h_{i_1}}, \dots, \vec{h_{i_p}})\}.$$

By Ramsey's Theorem, there is an infinite $S \subseteq \mathbb{N}^{>N}$ such that for every $i_1, \dots, i_p$, $j_1, \dots, j_p \in S$ with $i_1 < \dots < i_p$ and $j_1 < \dots < j_p$ we have

$$X(i_1, \dots, i_p) = X(j_1, \dots, j_p).$$

Now take a subset $J \subseteq S$ of size $|I|$. Let ϑ be the $\mathcal{L}(U)$-formula obtained as the conjunction of the following formulas with $\psi \in \Lambda$ and $i_1, \dots, i_p, j_1, \dots, j_p \in J$ such that $i_1 < \dots < i_p$ and $j_1 < \dots < j_p$:

$$\psi(\vec{y}_{i_1}, \dots, \vec{y}_{i_p}) \leftrightarrow \psi(\vec{y}_{j_1}, \dots, \vec{y}_{j_p}) \wedge \bigwedge_{n \in J} f(x_1, \dots, x_N, \vec{y}_n) = x_n \wedge U(\vec{y}_n).$$

By the above construction, we have that

$$\mathbb{M} \models \vartheta(a_1, \dots, a_N, (a_j)_{j \in J}, (\vec{h_j})_{j \in J}).$$

and thus

$$\mathbb{M} \models \exists \vec{y}_1 \dots \exists \vec{y}_{|J|} \ \vartheta(a_1, \dots, a_N, (a_j)_{j \in J}, (\vec{y_j})_{j \in J}).$$

Since $(a_n)_{n \in \mathbb{N}}$ is an indiscernible sequence, we have that

$$\mathbb{M} \models \exists \vec{y}_1 \dots \exists \vec{y}_{|J|} \ \vartheta(a_1, \dots, a_N, (a_n)_{n \in I}, (\vec{y_j})_{j \in J}).$$

Hence $\Delta_1 \cup \Delta_2$ is satisfiable in $\mathbb{M}$ and Σ is finitely satisfiable in $\mathbb{M}$.

□

2. Proof of Theorem 1.3

In this section $\mathcal{A} = (A, \dots)$ and $B \subseteq A$ are as in the setting of Theorem 1.3. Also let $\mathbb{M}$ be a monster model of T_B. We write $U(\mathbb{M})$ for the interpretation of the predicate U in the monster model $\mathbb{M}$.

In the rest of this paper, whenever we write dcl, we always refer to the definable closure with respect to the language $\mathcal{L}$. Remember that this definable closure gives a pregeometry. Throughout by dcl-*independent* we mean independent with respect to that pregeometry, likewise for an $\mathcal{L}$-definable subset S of $\mathbb{M}^n$, $\dim(S)$ denotes the dimension of S in that pregeometry.

The next result will be the main tool in proving Theorem 1.3.

Proposition 2.1. Suppose that

(1) for every formula $\varphi(\vec{x},\vec{y})$, indiscernible sequence $(\vec{g}_n)_{n\in\mathbb{N}}$ from $U(\mathbb{M})^m$ and $\vec{b}\in\mathbb{M}^p$, the set $\{n \mid \mathbb{M}\models\varphi(\vec{g_n},\vec{b})\}$ is finite or co-finite,

(2) for every formula $\varphi(x,\vec{y})$, indiscernible sequence $(a_n)_{n\in\mathbb{N}}$ from $\mathbb{M}$ and $\vec{b}\in\mathbb{M}^p$ with $a_n\notin \mathrm{dcl}(U(\mathbb{M}),\vec{b})$ for every n, the set $\{n \mid \mathbb{M}\models\varphi(a_n,\vec{b})\}$ is finite or co-finite.

Then T_B is dependent.

Proof: Let $(a_n)_{n\in\mathbb{N}}$ be an indiscernible sequence, $\varphi(x,\vec{y})$ be an $\mathcal{L}(U)$-formula and $\vec{b}\in\mathbb{M}^p$. We distinguish two cases.

Case I: $\{a_n : n\in\mathbb{N}\}$ is dcl-dependent over $U(\mathbb{M})$.
Using Lemma 1.6 take a function $f:\mathbb{M}^m\to\mathbb{M}$ that is $\mathcal{L}$-definable over $U(\mathbb{M})\cup\{a_1,\dots,a_N\}$ and an indiscernible sequence $(\vec{g}_n)_{n>N}$ from $U(\mathbb{M})^m$ such that for every $n>N$

$$f(\vec{g}_n)=a_n. \tag{2.1}$$

By (1), the set

$$\{n\ : n>N \text{ and } \ \mathbb{M}\models\varphi(f(\vec{g_n}),\vec{b})\}$$

is finite or cofinite. Hence also the set

$$\{n\in\mathbb{N}\ :\ \mathbb{M}\models\varphi(a_n,\vec{b})\}$$

is finite or cofinite.

Case II: $\{a_n : n\in\mathbb{N}\}$ is dcl-independent over $U(\mathbb{M})$.

Suppose there is an infinite set $S\subseteq\mathbb{N}$ such that for every $n\in S$, $a_n\in\mathrm{dcl}(U(\mathbb{M}),\vec{b})$. But then

$$\dim(U(\mathbb{M})\cup\{a_n\ :\ n\in S\})\le\dim(U(\mathbb{M}),\vec{b}),$$

which is impossible, since the first term is infinite and the second term is finite. Hence the statement follows from (2).

□

Now we are in a position to prove Theorem 1.3. For completeness we restate it.

Theorem 2.2. *Suppose that for every model $(\mathcal{M},N)$ of T_B the following hold:*

(i) *every subset of N^n definable in $(\mathcal{M},N)$ is a boolean combination of sets of the form $S\cap K$, where $S\subseteq M^n$ is definable in $\mathcal{M}$ and $K\subseteq M^n$ is $\emptyset$-definable in $(\mathcal{M},N)$,*

(ii) *every subset of M^m definable in $(\mathcal{M},N)$ is a boolean combination of subsets of M^m defined by*

$$\exists y_1\cdots\exists y_n\ U(y_1)\wedge\cdots\wedge U(y_n)\wedge\varphi(x_1,\dots,x_m,y_1,\dots,y_n),$$

where φ is a quantifier-free $\mathcal{L}$-formula (possibly with parameters from M),

(iii) *$(\mathcal{M},N)$ has o-minimal open core.*

Then T_B is dependent.

Proof: Let $\psi(\vec{x},\vec{y})$ be an $\mathcal{L}(U)$-formula, $(\vec{a}_n)_{n\in\mathbb{N}}$ an indiscernible sequence from $\mathbb{M}^m$, $\vec{b}\in\mathbb{M}^p$. We use Proposition 2.1 to conclude that the set

$$\{n:\mathbb{M}\models\psi(\vec{a}_n,\vec{b})\}$$

is either finite or cofinite. First consider the case that $\vec{a}_n\in U(\mathbb{M})$ for every n.

By (i), we can assume that there are an $\mathcal{L}$-formula $\varphi(\vec{x},\vec{y},\vec{z})$, an $\mathcal{L}(U)$-formula $\chi(\vec{x})$ without parameters and a tuple $\vec{c}\in\mathbb{M}^l$ such that

$$\mathbb{M}\models\psi(\vec{a_n},\vec{b}) \text{ iff } \mathbb{M}\models\varphi(\vec{a_n},\vec{b},\vec{c})\wedge\chi(\vec{a_n}).$$

Since o-minimal theories are dependent and $(\vec{a_n})_n$ is an indiscernible sequence, we have that $\{n \mid \mathbb{M}\models\psi(\vec{a_n},\vec{b})\}$ is finite or cofinite.

Now consider the case that $m=1$ and $a_n\notin \operatorname{dcl}\big(U(\mathbb{M}),\vec{b}\big)$ for every n. By (ii), we can assume that there is an $\mathcal{L}$-formula $\varphi(x,\vec{y})$ such that $\psi(x,\vec{y})$ is equivalent to

$$\exists z_1\ldots\exists z_m(\bigwedge_{n=1}^{m}U(z_n)\wedge\varphi(x,\vec{y},z_1,\ldots,z_m)).$$

Then for every $g_1,\ldots,g_m\in U(\mathbb{M})$ the set

$$\{a\in\mathbb{M}\ :\ \mathbb{M}\models\varphi(a,\vec{b},g_1,\ldots,g_m)\}$$

is a union of intervals and points. Since each a_n is dcl-independent from $\vec{b},g_1,\ldots,g_m$ we can assume that it is a finite union of open intervals and hence open. This implies that the set

$$\{z\in\mathbb{M}\ :\ \mathbb{M}\models\psi(z,\vec{b})\}$$

is a union of open sets and hence open. By (iii), it is a finite union of open intervals. Hence the set

$$\{n:\mathbb{M}\models\psi(\vec{a}_n,\vec{b})\}$$

is either finite or cofinite.

□

3. Dense pairs of o-minimal structures

In this section we prove that dense pairs of o-minimal structures as defined below are dependent. In the setting we are interested in, these structures are defined and studied for the first time by L. van den Dries in [vdD98], where it is shown that these pairs satisfy the conditions (i) and (ii) of Theorem 1.3. The theorems we rephrase below are from that paper.

Let T be a complete o-minimal theory expanding the theory of ordered abelian groups with a distinguished positive element 1. Let $\mathcal{L}$ be the language of T and $\mathcal{L}(U)$ as before. A pair $(\mathcal{M},\mathcal{N})$ of models of T is called a *dense pair* if $\mathcal{N}\preceq\mathcal{M}$, $M\neq N$ and N is dense in M. Let T_d be theory of of such pairs $(\mathcal{M},\mathcal{N})$ in the language $\mathcal{L}(U)$. The first result is basically condition (ii) of Theorem 1.3.

Theorem 3.1. *Each $\mathcal{L}$-formula in variables $\vec{y}=(y_1,\ldots,y_n)$ is T_d-equivalent to a boolean combination of formulas of the form*

$$\exists x_1\ldots\exists x_m\big(\bigwedge_{i=1}^{m}U(x_i)\wedge\varphi(\vec{x},\vec{y})\big),$$

where $\varphi(\vec{x},\vec{y})$ is an $\mathcal{L}$-formula.

The condition (i) is satisfied by dense pairs of o-minimal structure in a stronger form as follows.

Theorem 3.2. *Let $(\mathcal{M}, \mathcal{N})$ be a dense pair and $Y \subseteq N^n$. Then Y is definable in $(\mathcal{M}, \mathcal{N})$ if and only if there is a subset Z of M^n definable in $\mathcal{M}$ such that $Y = Z \cap N^n$.*

The following result gives condition (iii) when T is an o-minimal extension of the theory of the real field.

Theorem 3.3. *Let $(\mathcal{M}, \mathcal{N})$ be a dense pair. Suppose that $\mathcal{M}$ expands the real field. Then an open subset of $\mathbb{R}^n$ definable in $(\mathcal{M}, \mathcal{N})$ is definable in $\mathcal{M}$.*

As mentioned in [DMS08], condition (iii) holds for arbitrary dense pairs using Corollary 4.6 of [vdD98] and Theorem 4.14 of [DMS08]. Hence we have our desired consequence.

Corollary 3.4. Theories of dense pairs are dependent.

4. Groups with the Mann property

In this section $\mathcal{L}$ is the language of ordered rings and $\mathbb{R}$ denotes both the (ordered) real field in that language and its underlying set.

Let Γ be a dense subgroup of $\mathbb{R}^{>0}$. We say that Γ has the *Mann property* if for every $a_1, \dots, a_n \in \mathbb{Q}^\times$, there are only finitely many $(\gamma_1, \dots, \gamma_n) \in \Gamma^n$ such that $a_1\gamma_1 + \dots + a_n\gamma_n = 1$ and $\sum_{i \in I} a_i\gamma_i \neq 0$ for every proper nonempty subset I of $\{1, \dots, n\}$.

In the rest of this section we assume that Γ has the Mann property and $\Gamma/\Gamma^{[p]}$ is finite for every prime p, where $\Gamma^{[p]} := \{\gamma^p : \gamma \in \Gamma\}$. Let $\mathcal{L}(U; \Gamma)$ be the language $\mathcal{L}(U)$ augmented by a name for each element of Γ, and let $T(\Gamma)$ be the theory of $\big(\mathbb{R}, \Gamma, (\gamma)_{\gamma\in\Gamma}\big)$ in that language. Note that if $(R, G, (\gamma)_{\gamma\in\Gamma})$ is a model of $T(\Gamma)$, then R has a copy of $\mathbb{Q}(\Gamma)$ and Γ is a pure subgroup of G. From now on we denote models of $T(\Gamma)$ by (R, G) rather than $\big(R, G, (\gamma)_{\gamma\in\Gamma}\big)$.

Conditions (ii) and (iii) of Theorem 1.3 are given by the following two results.

Theorem 4.1. *(*[vdDG06]*, Theorem 7.5) Let (R, G) be a model of $T(\Gamma)$. Then every $\mathcal{L}(U)$-definable subset of R^m is a boolean combination of subsets defined by*

$$\exists y_1 \dots \exists y_n \big(U(y_1) \wedge \dots \wedge U(y_n) \wedge \varphi(x_1, \dots, x_m, y_1, \dots, y_n)\big),$$

where φ is a quantifier-free $\mathcal{L}$-formula (possibly with parameters).

Theorem 4.2. *(*[BEG07]*, Corollary 58) Let (R, G) be a model of $T(\Gamma)$. Then the open core of (R, G) is o-minimal.*

For a multiplicative group G, a tuple $\vec{k} = (k_1, \dots, k_n)$ of integers, and $m > 0$ let

$$G_{m,\vec{k}} := \{(g_1, \dots, g_n) \in G^n : g_1^{k_1} \cdots g_n^{k_n} \in G^{[m]}\},$$

a subgroup of the product group G^n.

¿From the assumption that $\Gamma/\Gamma^{[p]}$ is finite, it follows that $\Gamma_{m,\vec{k}}$ is of finite index in Γ^n for every $m, n > 0$ and $\vec{k} \in \mathbb{Z}^n$. Hence $G_{m,\vec{k}}$ is of finite index in G^n for every $m, n > 0$, $\vec{k} \in \mathbb{Z}$ whenever (R, G) is a model of $T(\Gamma)$. Moreover in that case we may choose coset representatives for $G_{m,\vec{k}}$ in G^n from Γ^n. Therefore condition (i) of Theorem 1.3 follows from the the next statement.

Theorem 4.3. *([BEG07], Proposition 53) Let (R,G) be a model of $T(\Gamma)$. A subset of G^n definable in (R,G) is a boolean combination of sets of the form $F \cap \vec{\gamma}G_{m,\vec{k}}$, where F is a semialgebraic subset of R^n, $m>0$, $\vec{k} \in \mathbb{Z}^n$, and $\vec{\gamma} \in \Gamma^n$.*

We get the desired result as a consequence of the above.

Corollary 4.4. The theory $T(\Gamma)$ is dependent.

4.1. **Groups with the Mann property and power functions.** Consider the structure

$$\widetilde{\mathbb{R}} := (\mathbb{R}, +, \cdot, 0, 1, x^\tau)$$

where $\tau \in \mathbb{R} \setminus \mathbb{Q}$ and x^τ is the unary function

$$x \mapsto \begin{cases} x^\tau, & x > 0, \\ 0, & x \leq 0. \end{cases}$$

Let $\mathcal{L}$ be its language and T^τ be the theory of $\widetilde{\mathbb{R}}$. Also let Γ be a dense subgroup of $\mathbb{R}^{>0}$ which has the Mann property and is a subset of the real algebraic numbers. Further suppose that $\Gamma/\Gamma^{[p]}$ be finite for prime number p. Now consider the $\mathcal{L}(U)$-structure $(\widetilde{\mathbb{R}}, \Gamma)$ and let $T^\tau(\Gamma)$ be its theory. In [H08], it is shown that for co-countably many τ all the assumptions of Theorem 1.3 hold.

Corollary 4.5. For co-countably many $\tau \in \mathbb{R} \setminus \mathbb{Q}$, the theory $T^\tau(\Gamma)$ is dependent.

5. The discrete case

Let T be a complete o-minimal theory extending the theory of real closed fields and $\mathcal{L}$ its language. Suppose also that T has quantifier elimination. In this section, we extend $\mathcal{L}$ to $\mathcal{L}(\phi)$ by adding a unary function symbol ϕ. Let $T(\phi)$ be a complete $\mathcal{L}(\phi)$-theory extending T and $\mathbb{M}$ a monster model of $T(\phi)$.

Whenever $\mathcal{A}, \mathcal{B}$ are models of T with $B \subseteq A$ and $X \subseteq A$, we let $\mathcal{B}\langle X \rangle$ denote the elementary substructure of $\mathcal{A}$ generated by B and X. If $X = \{x_1, \dots, x_n\}$ is a finite set then we write $\mathcal{B}\langle x_1, \dots, x_n \rangle$, rather than $\mathcal{B}\langle X \rangle$.

First we fix the following notation.

Definition 5.1. Let $(\mathcal{A}, \phi) \models T(\phi)$ and $C \subseteq A$. We define the ϕ-*closure* of C as

$$C^\phi := \{\ t(c_1, \dots, c_n)\ :\ c_1, \dots, c_n \in C, t \text{ is an } \mathcal{L}(\phi)\text{-term}\ \}.$$

Note that the ϕ-closure is obviously closed under ϕ and by quantifier elimination for T, it is an elementary substructure of $\mathcal{A}$.

Now we restate and prove Theorem 1.4.

Theorem 5.2. *Suppose that the following conditions hold.*

(i) *The theory $T(\phi)$ has quantifier elimination.*
(ii) *For every $(\mathcal{M}, \phi) \models T(\phi)$, $\mathcal{N} \preceq \mathcal{M}$ with $\phi(N) \subseteq N$ and every $m_1, \dots, m_n$ from M*

$$\dim \phi\big(N\langle m_1, \dots, m_n \rangle\big) - \dim \phi(N) \leq n,$$

(iii) *Let* $f : \mathbb{M}^{n+k} \to \mathbb{M}$ *and* $g : \mathbb{M}^{j+l}$ *be functions* $\emptyset$*-definable in the* $\mathcal{L}$*-reduct of* $\mathbb{M}$*,* $(\vec{a}_i)_{i\in\mathbb{N}}$ *an indiscernible sequence with* $a_{i,1}, \dots, a_{i,j} \in \phi(\mathbb{M})$ *for every* i*, and* $\vec{b_1} \in \mathbb{M}^k$ *and* $\vec{b_2} \in (\phi(\mathbb{M}))^l$*. Then the set*

$$\left\{ i \in \mathbb{N} \ : \ \mathbb{M} \models \phi(f(\vec{a}_i, \vec{b_1})) = g(a_{i,1}, \dots, a_{i,j}, \vec{b_2}) \right\}$$

is finite or cofinite.

Then $T(\phi)$ *is dependent.*

Proof: By (i), we just need to show that for every quantifier-free $\mathcal{L}(\phi)$-formula ψ, indiscernible sequence $(\vec{a_i})_{i\in\mathbb{N}}$ and tuple $\vec{b}$ from $\mathbb{M}$

$$J := \{ i \in \mathbb{N} \ : \ \mathbb{M} \models \psi(\vec{a_i}, \vec{b}) = 0 \} \text{ is finite or cofinite.} \tag{5.1}$$

We will prove this by induction on the number n of times ϕ occurs in ψ. If $n = 0$, this follows just from the dependency of o-minimal theories. Suppose (5.1) holds for all quantifier-free $\mathcal{L}(\phi)$-formulas in which ϕ occurs less than n times. Let ψ be a quantifier-free $\mathcal{L}(\phi)$-formula such that ϕ occurs n times in ψ. For a contradiction, let $(\vec{a_i})_{i\in\mathbb{N}}$ be an indiscernible sequence and $\vec{b} \in \mathbb{M}^l$ such that (5.1) does not hold for ψ. Since $n > 0$ and ψ is quantifier-free, take a function $f : \mathbb{M}^{n+l} \to \mathbb{M}$ that is $\emptyset$-definable in the $\mathcal{L}$-reduct of $\mathbb{M}$ and such that the term $\phi(f(\vec{a_i}, \vec{b}))$ occurs in $\psi(\vec{a_i}, \vec{b})$. By Theorem 1.5, we can assume that there is a set I with $I \subseteq J \subset \mathbb{N}$ such that $(\vec{a_i})_{i\in I}$ is an indiscernible sequence over $\vec{b}$. Now let A be the ϕ-closure of $\{\vec{a_i} : i \in I\}$. By (ii), there are $r_1, \dots, r_l \in \phi(A\langle b_1, \dots, b_n\rangle)$ such that

$$\phi(A\langle\vec{b}\rangle) = \phi(A)\langle r_1, \dots, r_l\rangle.$$

Then for every $j \in \mathbb{N}$ we have

$$\phi(f(\vec{a_j}, \vec{b})) \in \phi(A)\langle r_1, \dots, r_l\rangle.$$

Because $(\vec{a_i})_{i\in I}$ is an indiscernible sequence over $\vec{b}$, this implies that there exist natural numbers k and v with $1 \le v \le k$ and a function $g : \mathbb{M}^{m+l} \to \mathbb{M}$ that is $\emptyset$-definable in the $\mathcal{L}$-reduct of $\mathbb{M}$ such that for every increasing sequence $i_1 < \dots < i_k$ of elements of I

$$\phi(f(\vec{a_{i_v}}, \vec{b})) = g(t_1(\vec{a_{i_1}}, \dots, \vec{a_{i_k}}), \dots, t_m(\vec{a_{i_1}}, \dots, \vec{a_{i_k}}), r_1, \dots, r_l). \tag{5.2}$$

Take an infinite subset $\mathscr{K}$ of the set of k-element subsets of I and take an infinite subset $\mathscr{L}$ of the set of k-element subsets of $\mathbb{N} \setminus J$ such that for every $S, T \in \mathscr{K} \cup \mathscr{L}$ either $s < t$ for every $s \in S$, $t \in T$ or $t < s$ for every $s \in S$, $t \in T$.

Since $(\vec{a_i})_{i\in\mathbb{N}}$ is indiscernible, the sequence

$$(\vec{a_{i_1}}, \dots, \vec{a_{i_k}}, t_1(\vec{a_{i_1}}, \dots, \vec{a_{i_k}}), \dots, t_m(\vec{a_{i_1}}, \dots, \vec{a_{i_k}}))_{\{i_1,\dots,i_k\}\in\mathscr{K}\cup\mathscr{L}}$$

is indiscernible as well. By (5.2), the equation

$$\phi(f(\vec{a_{i_v}}, \vec{b})) = g(t_1(\vec{a_{i_1}}, \dots, \vec{a_{i_k}}), \dots, t_m(\vec{a_{i_1}}, \dots, \vec{a_{i_k}}), r_1, \dots, r_l)$$

holds infinitely many element of this sequence. Because of (iii), we get that this equations actually holds for cofinitely many elements of the sequence. By substituting (5.2) in ψ, we get a quantifier-free $\mathcal{L}(\phi)$-formula ψ' in which ϕ occurs less than n times such that

$$\mathbb{M} \models \psi(\vec{a_{i_v}}, \vec{b}) \leftrightarrow \psi'(\vec{a_{i_1}}, \dots, \vec{a_{i_k}}, t_1(\vec{a_{i_1}}, \dots, \vec{a_{i_k}}), \dots, t_m(\vec{a_{i_1}}, \dots, \vec{a_{i_k}}), \vec{b}, r_1, \dots, r_l)$$

holds for cofinitely many $\{i_1,\dots,i_k\} \in \mathscr{K} \cup \mathscr{L}$. But

$$\mathbb{M} \models \psi(\vec{a_{i_v}},\vec{b}) \text{ iff } \{i_1,\dots,i_v,\dots,i_k\} \in \mathscr{K} \text{ for some } i_1,\dots,i_k.$$

Hence the set of $\{i_1,\dots,i_k\} \in \mathscr{K} \cup \mathscr{L}$ such that

$$\mathbb{M} \models \psi'(\vec{a_{i_1}},\dots,\vec{a_{i_k}},t_1(\vec{a_{i_1}},\dots,\vec{a_{i_k}}),\dots,t_m(\vec{a_{i_1}},\dots,\vec{a_{i_k}}),\vec{b},r_1,\dots,r_l)$$

is neither finite nor cofinite in $\mathscr{K} \cup \mathscr{L}$. This contradicts the induction hypothesis.

□

6. Tame pairs

In this section, we consider tame pairs of o-minimal structures. Dolich, Miller and Steinhorn asked in [DMS08] whether tame pairs of o-minimal structures are dependent. Using Theorem 5.2 we give a positive answer here.

Let T be a complete o-minimal theory extending the theory of real closed fields and $\mathcal{L}$ its language. After extending T by definitions, we can assume without loss of generality that T has quantifier elimination. Let T_t be the theory of all structures $(\mathcal{A},B)$, where $\mathcal{A},\mathcal{B}$ are models of T such that $A \neq B$, $\mathcal{B} \preceq \mathcal{A}$ and for every $a \in A$ which is in the convex hull of B, there is a unique $\phi(a) \in B$ such that $|a-\phi(a)| < b$ for all $b \in B^{>0}$. Note that ϕ can be extended to all A by setting $\phi(a) = 0$ if a is not in the convex hull of B, and we call the resulting map as the *standard part map*. The structures $(\mathcal{A},B)$ and (A,ϕ) are interdefinable. Since T has quantifier elimination, it follows from the results from [vdDL95] that T_t is complete and $(\mathcal{A},\phi)$ has quantifier elimination.

The theory T_t of tame pairs of models of T is closely related to the theory T_c of pairs $(\mathcal{A},V)$, where $\mathcal{A} \models T$ and V is a T-convex subring of A and $V \neq A$ (here and below a *T-convex* subring of a model $\mathcal{A}$ of T is a convex subring that is closed under all the continous $\emptyset$-definable unary functions). Since T_c is weakly o-minimal (see [vdDL95]), T_c is dependent. Note that for every model $(\mathcal{A},\mathcal{B})$ of T_t, the pair $(\mathcal{A},V)$ is a model of T_c, where V is the convex closure of B. In this case, for every $b \in B$ and $a \in A$

$$\phi(a) = b \text{ iff } a = b \text{ or } (a-b \in V \text{ and } (a-b)^{-1} \notin V) \text{ or } (b = 0 \text{ and } a \notin V). \tag{6.1}$$

Since V is local subring of A, there is a valuation $v : A^{\times} \to \Gamma$ with

$$V = \{a \in A \ : \ v(a) \geq 0\}.$$

Further let $\bar{V}$ be the residue field of v. By Theorem 2.12 of [vdDL95] the map $B \hookrightarrow V \to \bar{V}$ is bijective. Thus $\bar{V}$ can be made into to a model of T such that the map becomes an isomorphisms of $\mathcal{L}$-structures. We need the following result from [vdDL95].

Theorem 6.1. *Let $\mathcal{A},\mathcal{A}^*$ be models of T such that $\mathcal{A} \preceq \mathcal{A}^*$ and let $a \in A^* \setminus A$. Suppose that W and W' are T-convex subrings of $\mathcal{A}$ and $\mathcal{A}\langle a\rangle$ with $A \cap W' = W$ and $\bar{a} \in \overline{W'} \setminus \overline{W}$. Then $\overline{W'} = \overline{W}\langle\bar{a}\rangle$.*

Now we are in a position to prove our main result.

Theorem 6.2. *T_t is dependent.*

Proof: We will show that T_t satisfies the assumptions of Theorem 5.2. As already mentioned above, T_t satisfies (i) by Theorem 5.9 from [vdDL95]. Now we consider (iii). Let $(\vec{a}_i)_{i\in\mathbb{N}}$ be an indiscernible sequence from $\mathbb{M}^n$ such that there is a positive $j \leq n$ with $a_{i,1},\dots,a_{i,j} \in \phi(\mathbb{M})$ for every $i \in \mathbb{N}$. Also let $\vec{b_1} \in \mathbb{M}^k$ and $\vec{b_2} \in \phi(\mathbb{M})^l$ and further f, g be as in (iii) of Theorem 5.2. We now want to show that

$$J := \{i \in \mathbb{N} \ : \ \mathbb{M} \models \phi(f(\vec{a}_i, \vec{b_1})) = g(a_{i,1},\dots,a_{i,j},\vec{b_2})\}$$

is finite or cofinite. Since $\phi(\mathbb{M})$ is a model of T, we have that for every $i \in \mathbb{N}$

$$g(a_{i,1},\dots,a_{i,j},\vec{b_2}) \in \phi(\mathbb{M}).$$

By (6.1), there is an $\mathcal{L}(U)$-formula ψ such that

$$T_t \models \phi(f(\vec{a}_i, \vec{b_1})) = g(a_{i,1},\dots,a_{i,j},\vec{b_2}) \in \phi(\mathbb{M}) \Longleftrightarrow T_c \models \psi(a_{i,1},\dots,a_{i,j},\vec{b_2}).$$

Since T_c is dependent, J is finite or cofinite.

It is left to show (ii). So let $(\mathcal{A},\phi) \models T_t$ and $\mathcal{B} \preceq \mathcal{A}$ with $\phi(B) \subseteq B$ and $a_1,\dots,a_n \in A$. Let W_B be the T-convex closure of $\phi(B)$ in B and W_A the T-convex closure of $\phi(A)$ in A. As noted before there is an isomorphism ι from $\phi(A)$ to $\overline{W_A}$ mapping $\phi(a)$ to $\overline{\phi(a)}$ for every $a \in A$. Without loss of generality, we can assume that there is a positive $l \leq n$ such that $a_1,\dots,a_l \in W_A$ and for every function $f : A^n \to A$ $\emptyset$-definable in $\mathcal{A}$

$$\text{if } f(a_1,\dots,a_n) \notin \langle a_1,\dots,a_l\rangle, \text{ then } f(a_1,\dots,a_n) \notin W_A. \tag{6.2}$$

Using Theorem 6.1 inductively, we get that $\overline{B\langle a_1,\dots,a_l\rangle \cap W_A} = \overline{W_B}\langle \bar{a}_1,\dots,\bar{a}_l\rangle$. Now by (6.2)

$$B\langle a_1,\dots,a_l\rangle \cap W_A = B\langle a_1,\dots,a_n\rangle \cap W_A.$$

Therefore

$$\overline{B\langle a_1,\dots,a_n\rangle \cap W_A} = \overline{W_B}\langle \bar{a}_1,\dots,\bar{a}_l\rangle.$$

Hence $\dim \overline{B\langle a_1,\dots,a_n\rangle \cap W_A}/\overline{W}_B \leq n$. Using the isomorphism ι, we get that

$$\dim \phi(B\langle a_1,\dots,a_n\rangle) - \dim \phi(B) \leq n.$$

So assumption (ii) of Theorem 5.2 also holds for T_t.

□

7. Discrete groups

Let $\tilde{\mathbb{R}}$ be an o-minimal expansion of $(\mathbb{R},<,+,\cdot,0,1)$ which is polynomially-bounded with field of exponents $\mathbb{Q}$. Let T be the theory of $\tilde{\mathbb{R}}$ and $\mathcal{L}$ be its language. We consider the structure $(\tilde{\mathbb{R}}, 2^{\mathbb{Z}})$. Since $2^{\mathbb{Z}}$ is discrete, we can define the following function $\lambda : \mathbb{R} \to 2^{\mathbb{Z}} \cup \{0\}$ by

$$\lambda(x) := \begin{cases} g, & x > 0, g \in 2^{\mathbb{Z}} \text{ and } g \leq x < 2g; \\ 0, & x < 0. \end{cases}$$

Again, it is easy to see that the structures $(\tilde{\mathbb{R}}, 2^{\mathbb{Z}})$ and $(\tilde{\mathbb{R}}, \lambda)$ are interdefinable. In [M05] generalizing the results from [vdD85], it is shown that the latter has quantifier elimination up to $\tilde{\mathbb{R}}$. For the following, we can assume that $\tilde{\mathbb{R}}$ has quantifier elimination. Let T_{disc} be the theory of $(\tilde{\mathbb{R}}, 2^{\mathbb{Z}})$ and let $\mathcal{L}(\lambda)$ be the extension of $\mathcal{L}$ by a function symbol for the map λ.

One of the key properties of polynomially-bounded o-minimal structures is the *Valuation Inequality*. We state here a particular case for theories with field of exponents $\mathbb{Q}$. For the general statement, see Corollary 5.6 in [vdD97].

Theorem 7.1. *Let $\mathcal{M}$, $\mathcal{N}$ be models of T with $\mathcal{N} \preceq \mathcal{M}$ and let $x_1, \dots, x_n \in \mathcal{M}$. Then there are $y_1, \dots, y_n \in \mathcal{M}$ such that for every $z \in \mathcal{N}\langle x_1, \dots, x_n\rangle$, there are $q_1, \dots, q_n \in \mathbb{Q}$ and there is $c \in \mathcal{N}$ with*

$$1 \leq \frac{z}{c \cdot y_1^{q_1} \dots y_n^{q_n}} < 2$$

Corollary 7.2. Let $\mathcal{M}, \mathcal{N}$ be as in the previous theorem with $(\mathcal{M}, G) \models T_{disc}$ and $\mathcal{N}$ is closed under λ. Let $x_1, \dots, x_n \in \mathcal{M}$. Then there are $y_1, \dots, y_n \in G$ such that for every $z \in \mathcal{N}\langle x_1, \dots, x_n\rangle$, there are $q_1, \dots, q_n \in \mathbb{Q}$ and there is $g \in \mathcal{N} \cap G$ with

$$\lambda(z) = g \cdot y_1^{q_1} \dots y_n^{q_n}.$$

We fix the following notation: Let G be an abelian group and $n > 0$. Then $G^{[n]} := \{g^n : g \in G\}$. For a pure subgroup H of G and $g_1, \dots, g_n \in G^n$, we define $H_G\langle g_1, \dots, g_n\rangle$ to be the smallest pure subgroup

$$\left\{(h \cdot {g_1}^{k_1} \cdots {g_n}^{k_n})^{\frac{1}{m}} \mid h \in H, k_1, \dots, k_n \in \mathbb{Z}, m > 0, h \cdot {g_1}^{k_1} \cdots {g_n}^{k_n} \in G^{[m]}\right\}$$

of G containing H and $g_1, \dots, g_n$.

Corollary 7.3. Let $(M, G) \preceq (M^*, G^*) \models T_{disc}$ and $g \in G^* \setminus G$. Then

$$(M\langle \vec{g}\rangle, \lambda(M)_{G^*}\langle g\rangle) \preceq (M^*, G^*).$$

Proof: By Corollary 7.2, there is $\vec{y} \in M\langle \vec{g}\rangle$ such that $\lambda(M\langle \vec{g}\rangle) \subseteq G_{G^*}\langle \vec{y}\rangle$. Since $\lambda(\vec{g}) = \vec{g}$ and M is closed under λ, we can replace $\vec{y}$ by $\vec{g}$.

□

Theorem 7.4. *T_{disc} is dependent.*

Proof: Again, we need just to check that T_{disc} satisfies the assumption of Theorem 5.2. Quantifier elimination is shown in the proof of Theorem 3.4.2 in [M05]. Assumption (ii) follows directly from Corollary 7.2. It is only left to show (iii). Therefore let $\mathbb{M} = (M, G)$ be a monster model of T_{disc} and take an indiscernible sequence $(\vec{a}_i)_{i \in \mathbb{N}}$ from $\mathbb{M}^n$ such that there is a positive $j \leq n$ with $a_{i,1}, \dots, a_{i,j} \in \lambda(\mathbb{M})$. Further let $\vec{b_1} \in \mathbb{M}^k$ and $\vec{b_2} \in \lambda(\mathbb{M})^l$. For a contradiction, suppose that there are functions $f : \mathbb{M}^{n+k} \to \mathbb{M}$ and $g : \mathbb{M}^{j+l} \to \mathbb{M}$ that are $\emptyset$-definable in the $\mathcal{L}$-reduct of $\mathbb{M}$ such that

$$J := \{i \in \mathbb{N} \ : \ \mathbb{M} \models \lambda(f(a_{i,1}, \dots, a_{i,n}, \vec{b_1})) = g(a_{i,j}, \dots, a_{i,n}, \vec{b_2})\}$$

is neither finite nor cofinite. Again by Theorem 1.5, we can take a subsequence $(a_i)_{i \in I}$ which is indiscernible over $\vec{b_1}, \vec{b_2}$ and $I \subseteq J$. Hence for every $i \in I$ we have $g(a_{i,1}, \dots a_{i,j}, \vec{b_2}) \in G$. By Corollary 7.3, for every $i \in I$ there are $q_1, \dots, q_j \in \mathbb{Q}$ such that

$$g(a_{i,1}, \dots a_{i,j}, \vec{b_2}) = a_{i,1}^{q_1} \dots a_{i,j}^{q_j} \cdot \vec{b_2}^{\vec{k}},$$

where $\vec{k} \in \mathbb{Z}^l$ and $\vec{b_2}^{\vec{k}}$ is short for $b_{2,1}^{k_1} \cdots b_{2,l}^{k_l}$. Since T is o-minimal and hence dependent, this equation holds for cofinitely many $i \in \mathbb{N}$. Now we need to show that for cofinitely many $i \in \mathbb{N}$ $a_{i,1}^{q_1} \dots a_{i,j}^{q_j} \cdot \vec{b_2}^{\vec{k}} \in G$. Therefore let $m \in \mathbb{N}$ such that

$q_1 \cdot m, \dots, q_j \cdot m \in \mathbb{Z}$ and $m \cdot \vec{k} \in \mathbb{Z}^l$. So we need to show that for cofinitely many $i \in \mathbb{N}$, we have

$$a_{i,1}^{m\cdot q_1} \dots a_{i,j}^{m\cdot q_j} \in \vec{b_2}^{m\cdot \vec{k}} \cdot G^{[m]}. \tag{7.1}$$

$G^{[m]}$ has only finitely many cosets in G, since $|2^{\mathbb{Z}} : (2^{\mathbb{Z}})^{[m]})| = m$. Further $1, 2, \dots, 2^{m-1}$ are representatives of this cosets. Let $l \in \{0, \dots, m-1\}$ be such that $\vec{b_2}^{\vec{p}}$ is in $2^l \cdot G^{[m]}$. Then for every $i \in \mathbb{N}$, we have that (7.1) holds iff

$$a_{i,1}^{q_1} \dots a_{i,j}^{q_j} \in 2^l \cdot G^{[m]}. \tag{7.2}$$

Since (7.1) holds for $i \in J$ and $(a_i)_{i\in\mathbb{N}}$ is an indiscernible sequence, the condition (7.2) holds for all $i \in \mathbb{N}$. Hence for cofinitely many $i \in \mathbb{N}$, we have that $g(a_{i,1}, \dots a_{i,j}, \vec{b_2}) \in G$. By definition of λ, we further get that the assertion that

$$\mathbb{M} \models \lambda(f(\vec{a}_i, \vec{b_1})) = g(a_{i,1}, \dots a_{i,j}, \vec{b_2})$$

is equivalent to

$$\mathbb{M} \models 1 \leq \frac{f(\vec{a}_i, \vec{b_1})}{g(a_{i,1}, \dots a_{i,j}, \vec{b_2})} < 2.$$

Since T is dependent, the right hand side must hold for cofinitely many $i \in \mathbb{N}$. This a contradiction to J not being cofinite. So we have shown that assumption (iii) of Theorem 5.2 holds for T_{disc}.

□

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