A TALK ON MANN PAIRS

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In [2], we isolated the following property of the multiplicative subgroups of $\mathbb{C}$ of finite rank.

Definition 0.1. Let K be any field and G a subgroup of $K\times$. We say G has the *Mann property* if for every $k_1, \ldots, k_n \in \mathbb{Q}^\times$, there are only finitely many $(g_1, \ldots, g_n) \in G^n$ such that $\sum_{i=1}^n k_i g_i = 1$ and $\sum_{i \in I} k_i g_i \neq 0$ for every nonempty proper subset I of $\{1, \ldots, n\}$.

For an algebraically closed field K of characteristic zero, this turns out to be the same as the *Mordell-Lang property*, that is for every algebraic subset V of K^n, the intersection $G^n \cap V$ is a finite union of cosets of subgroups of G^n definable in the group G. This way we get a lot of model theoretic information on the structure (K, G). For instance, it is *near model complete*, which is to say that the subsets of K^n definable in the pair (K, G) are boolean combinations of existentially definable subsets. We also get that it is ω-stable when G is divisible, hence we have a notion of dimension, called *Morley rank*, different than the usual dimension; for instance the group G has Morley rank 1 and K has Morley rank ω. Morley rank somehow gives a finer information than the one given by the usual dimension.

Another examples of groups with this property is the group of exponentials of algebraic numbers. We use the following to show this:

Lemma 0.2. *Let K be a field with subfield E, and G and Γ are subgroups of $K^\times$ with $\Gamma \subseteq G$. Suppose Γ is a pure subgroup of G and each root of unity in G lies in Γ. Then the following two conditions are equivalent:*

1. *for any $c_1, ..., c_m \in E^\times$ the equation $c_1 x_1 + \cdots + c_m x_m = 1$ has the same nondegenerate solutions in Γ as in G;*
2. *whenever $g_1, ..., g_n \in G$ are multiplicatively independent over Γ, they are algebraically independent over $E(\Gamma)$.*

Now it follows from the Lindemann-Weierstrass theorem that $\exp(\mathbb{Q}^{\mathrm{ac}})$ satisfies the second property. As a matter of fact, such a proof shows even more. Namely it proves that for every $n > 0$, there is a finite subset $G(n) = \{(1, \ldots, 1)\}$ of G^n such that for every $k_1, \ldots, k_n \in \mathbb{Q}^\times$, the nondegenerate solutions of $k_1 x_1 + \cdots + k_n x_n = 1$ in G^n are in $G(n)$. Moreover we could take $k_1, \ldots, k_n$ from $\mathbb{Q}^{\mathrm{ac}}$. This gives us the following very uniform form of Mann/Mordell-Lang property.

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Definition 0.3. Let Ω be an ambient algebraically closed field and let K be a subfield of Ω and G a subgroup of $\Omega^\times$. We say that (K,G) is a *Mann pair* if for every $n>0$, there is a finite subset $G(n)$ of G^n such that for every $a_1,\dots,a_n\in K^\times$, the nondegenerate solutions of $a_1x_1+\cdots+a_nx_n=1$ in G^n lie in $G(n)$.

In our paper we prove the following.

Theorem 0.4. *Let K be algebraically closed and G is of finite rank with $K^\times\cap G=\{1\}$. Then (K,G) is a Mann pair.*

Allow me to illustrate this in a very simple case. Let K be an algebraically closed field and let G be the subgroup of $K(t)$ generated by $t-5$ and t^2+2t-7. Note that $G\cap K^\times=\{1\}$ and obviously G is finitely generated. Then according to our theorem (K,G) must be a Mann pair. However, in this case we could even tell what the possible solutions are, using the following theorem from [1].

Theorem 0.5. *Let K be of characteristic zero and let F be a function field over K. Also let g be the genus of $F|K$, and let S be a finite subset of $\mathcal{R}(F|K)$ and $n\geq 2$. Suppose $u_1,\dots,u_n$ are S-units and $(u_1,\dots,u_n)$ is a non-degenerate solution of $x_1+\cdots+x_n=0$. Then*

$$H(u_1,\dots,u_n)\leq \frac{1}{2}(n-1)(n-2)\{|S|+\max(0,2\,\mathrm{g}-2)\}.$$

In our case $F=K(t)$ and hence $H(u_1,\dots,u_n)=\max\{\deg_t u_1,\dots,\deg_t u_n\}$ and also the cardinality of S is at most 4. Thus we get that the degrees of the polynomials involved should be less than $\frac{1}{2}(n-1)(n-2)+4$.

On the model theory side of the story we have both "near model completeness" and "orthogonality". Namely subsets of Ω^n definable in the structure (Ω,K,G) are given by boolean combinations of sets of defined by

$$\exists y\exists z(y\in K^p\wedge z\in G^q\wedge\phi(x,y,z)),$$

and deifnable subsets of $K^m\times G^n$ are finite unions of $X\times Y$ where X is definable in the field K and Y is definable in the group G.

Also when K is algebraically closed and G is divisible, the structure (Ω,K,G) becomes ω-stable, which is to say that there is notion of dimension "somehow" refining the usual one. It is called the Morley rank. In this particular case $\mathrm{MR}(K)=\mathrm{MR}(G)=1$ and $\mathrm{MR}(\Omega)=\aleph_0$.

References

[1] W.D. Brownawell and D.W. Masser. Vanishing sums in function fields. *Math. Proc. Camb. Phil. Soc.*, 100:427–434, 1986.

[2] Lou van den Dries and Ayhan Günaydın. The fields of real and complex numbers with a small multiplicative group. *Proc. London Math. Soc. (3)*, 93(1):43–81, 2006.