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MANN PAIRS

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ABSTRACT. Mann proved in the 1960s that for any $n \geq 1$ there is a finite set E of n-tuples $(\eta_1, \ldots, \eta_n)$ of complex roots of unity with the following property: if $a_1, \ldots, a_n$ are any rational numbers and $\zeta_1, \ldots, \zeta_n$ are any complex roots of unity such that $\sum_{i=1}^n a_i\zeta_i = 1$ and $\sum_{i \in I} a_i\zeta_i \neq 0$ for all nonempty $I \subseteq \{1, \ldots, n\}$, then $(\zeta_1, \ldots, \zeta_n) \in E$. Taking an arbitrary field $\mathbf{k}$ instead of $\mathbb{Q}$ and any multiplicative group in an extension field of $\mathbf{k}$ instead of the group of roots of unity, this property defines what we call a Mann pair $(\mathbf{k}, \Gamma)$. We show that Mann pairs are robust in certain ways, construct various kinds of Mann pairs, and characterize them model-theoretically.

1. Introduction

Throughout this paper we use the following notations and conventions. We let Ω be an ambient algebraically closed field (taken to be $\mathbb{C}$ if suggested by the context), and let Γ be a subgroup of the multiplicative group $\Omega^\times$. We say that Γ has *finite rank* if Γ has a finitely generated subgroup Γ_0 such that Γ/Γ_0 is a torsion group. We let m, n range over $\mathbb{N} = \{0, 1, 2, \ldots\}$. For $r \in \Omega$ and $\vec{s} = (s_1, \ldots, s_n) \in \Omega^n$ we put $r\vec{s} := (rs_1, \ldots, rs_n) \in \Omega^n$. In particular, if $\alpha \in \Gamma$ and $\vec{\gamma} \in \Gamma^n$, then $\alpha\vec{\gamma} \in \Gamma^n$. We let $\mathbb{U} \subseteq \mathbb{C}^\times$ be the group of complex roots of unity, and $\mathbb{Q}^{\text{ac}} \subseteq \mathbb{C}$ the field of complex algebraic numbers.

Let $a_1, \ldots, a_n \in \Omega$, $n \geq 1$, and consider the equation

$$(*) \qquad a_1x_1 + \cdots + a_nx_n = 1.$$

A *solution* of $(*)$ is a tuple $(s_1, \ldots, s_n) \in \Omega^n$ such that $a_1s_1 + \cdots + a_ns_n = 1$; such a solution is said to be *nondegenerate* if $\sum_{i \in I} a_is_i \neq 0$ for all nonempty $I \subseteq \{1, \ldots, n\}$, and is said to be in Γ if $(s_1, \ldots, s_n) \in \Gamma^n$.

In [5] we defined Γ to have the *Mann property* if for all $n \geq 1$ and nonzero $a_1, \ldots, a_n$ in the prime field of Ω the equation $(*)$ has only finitely many non-degenerate solutions in Γ; equivalently, for all $n \geq 1$ and $a_1, \ldots, a_n \in \Omega^\times$ the equation $(*)$ has only finitely many non-degenerate solutions in Γ, see [5]. The Mann property is a Mordell-Lang type property. Mann [12] gave an effective proof that $\mathbb{U}$ has the Mann property. More generally, but less effectively, [6], [11], and [14] show that if Ω has characteristic zero and Γ has finite rank, then Γ has the Mann property. In [5] we studied the model theory of structures (Ω, Γ) when Γ has the Mann property. We now define a *uniform* version of the Mann property involving also a subfield of Ω. From now on $\mathbf{k}$ is a subfield of Ω.

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Call $(\mathbf{k},\Gamma)$ a *Mann pair* if for each $n \geq 1$ there is a finite $E \subseteq \Gamma^n$ such that for all $a_1,\dots,a_n \in \mathbf{k}^\times$ all non-degenerate solutions of $(*)$ in Γ lie in E.

Suppose $(\mathbf{k},\Gamma)$ is a Mann pair. It is obvious that then Γ has the Mann property and that if $\mathbf{k}'$ is a subfield of $\mathbf{k}$ and Γ' a subgroup of Γ, then $(\mathbf{k}',\Gamma')$ is also Mann pair. Taking $n=1$ in the definition we see that $\mathbf{k}^\times \cap \Gamma$ is finite, and so all elements of $\mathbf{k}^\times \cap \Gamma$ are roots of unity.

Actually, Theorem 1 of [12] implies that $(\mathbb{Q},\mathbb{U})$ is a Mann pair; conversely, Corollary 5.1 below says that if $(\mathbb{Q},\Gamma)$ is a Mann pair and $\Gamma \subseteq (\mathbb{Q}^{\mathrm{ac}})^\times$, then $\Gamma \subseteq \mathbb{U}$. Thus $(\mathbb{Q},(1+\sqrt{2})^{\mathbb{Z}})$ is not a Mann pair, although $(1+\sqrt{2})^{\mathbb{Z}} \subseteq (\mathbb{Q}^{\mathrm{ac}})^\times$ has the Mann property and $\mathbb{Q}^\times \cap (1+\sqrt{2})^{\mathbb{Z}} = \{1\}$. In Section 2 we indicate simple examples of Mann pairs, such as:

(1) $(\mathbb{Q}^{\mathrm{ac}},\exp(\mathbb{Q}^{\mathrm{ac}}))$ is a Mann pair (Lindemann's theorem).
(2) If $v : K^\times \to v(K^\times)$ is a valuation on a subfield K of Ω and $\mathbf{k}$ is a subfield of K and Γ a subgroup of $K^\times$ such that v is trivial on $\mathbf{k}$ and injective on Γ, then $(\mathbf{k},\Gamma)$ is a Mann pair.

A more substantial source of Mann pairs is the following:

Theorem 1.1. *Suppose Ω has characteristic zero, $\mathbf{k}$ is algebraically closed, $\mathbf{k}^\times \cap \Gamma = \{1\}$, and Γ has finite rank. Then $(\mathbf{k},\Gamma)$ is a Mann pair.*

This is an analogue for Mann pairs of the result from [6], [11], [14] that was mentioned earlier. In Section 3 we first deal with the special case that Γ is finitely generated and $\mathbf{k}(\Gamma)$ has transcendence degree 1 over $\mathbf{k}$; the key here is a height bound by Brownawell and Masser [3]. With this special case as a stepping stone and using various facts established in Sections 4 and 5 we prove Theorem 1.1 in Section 6. This proof is effective, at least in principle, but we do not elaborate this aspect in the present paper. Section 5 also describes some curious Mann pairs in positive characteristic as well as some Mann pairs $(\mathbb{Q}_p,\Gamma)$ obtained by Teichmüller lifting.

Next we consider the structure $(\Omega,\mathbf{k},\Gamma)$, that is, the algebraically closed field Ω with distinguished subsets $\mathbf{k}$ and Γ. In Sections 4 and 7 we show:

Theorem 1.2. *The following are equivalent:*

(1) $(\mathbf{k},\Gamma)$ *is a Mann pair;*
(2) *for each $n \geq 1$ the set $\{(x,y) \in \mathbf{k}^n \times \Gamma^n :\ x_1y_1 + \cdots + x_ny_n = 0\}$ is a finite union of sets $X \times Y$ where $X \subseteq \mathbf{k}^n$ is definable in the field $\mathbf{k}$ and $Y \subseteq \Gamma^n$ is definable in the group Γ;*
(3) *for all m,n, every subset of $\mathbf{k}^m \times \Gamma^n$ definable in $(\Omega,\mathbf{k},\Gamma)$ is a finite union of sets $X \times Y$ with $X \subseteq \mathbf{k}^m$ definable in the field $\mathbf{k}$ and $Y \subseteq \Gamma^n$ definable in the group Γ.*

Here and below, "definable" means "definable with parameters" unless we specify otherwise. Thus (3) says that the structure induced by $(\Omega,\mathbf{k},\Gamma)$ on $\mathbf{k}$ and Γ (using parameters from Ω) is just the field structure of $\mathbf{k}$ and the group structure of Γ, respectively, and that these two structures only interact trivially inside $(\Omega,\mathbf{k},\Gamma)$.

In Section 8 we prove that for Mann pairs $(\mathbf{k},\Gamma)$ with $[\Omega:\mathbf{k}] > 2$, the complete theory $\mathrm{Th}(\Omega,\mathbf{k},\Gamma)$ is determined by $\mathrm{Th}(\mathbf{k})$ and $\mathrm{Th}(\Gamma)$ after adding names for enough elements of $\mathbf{k}$ and Γ to witness that $(\mathbf{k},\Gamma)$ is a Mann pair. Under the additional assumption that $\mathbf{k}$ is algebraically closed we show that $(\Omega, \mathbf{k}, \Gamma)$ is stable, and we make further observations along these lines.

We let $\mathbf{k}^{\mathrm{ac}}$ be the algebraic closure in Ω of any subfield $\mathbf{k}$ of Ω.

Acknowledgement. The referee has pointed out that Theorem 1.1 also follows from Hrushovski-Pillay [7]. At the end of Section 6 we indicate how. The referee also mentioned that Sections 7 and 8 are related to earlier work by Casanovas and Ziegler [4] and by Pillay [13].

2. Some examples of Mann pairs

It will be useful to consider also *homogeneous* linear equations. Let $n \geq 1$ and let $a_1, \dots, a_n \in \Omega$. A *solution* of the homogeneous linear equation

$$a_1 x_1 + \cdots + a_n x_n = 0$$

is a tuple $(s_1, \dots, s_n) \in \Omega^n$ such that $a_1 s_1 + \cdots + a_n s_n = 0$; such a solution is said to be *non-degenerate* if $\sum_{i \in I} a_i s_i \neq 0$ for all nonempty proper subsets I of $\{1, \dots, n\}$, and is said to be in Γ if $(s_1, \dots, s_n) \in \Gamma^n$.

2.1. **Alternative definition of Mann pairs.** The equivalence of (1) and (2) in the next lemma expresses being a Mann pair in terms of homogeneous linear equations. Assuming $\mathbf{k}^\times \cap \Gamma$ is finite, the equivalence between (1) and (3) expresses $(\mathbf{k}, \Gamma)$ being a Mann pair purely in terms of the subgroup $G = \mathbf{k}^\times \Gamma$ of $\Omega^\times$ generated by $\mathbf{k}^\times$ and Γ.

Lemma 2.1. *Let $G = \mathbf{k}^\times \Gamma$. Then the following are equivalent:*

1. *$(\mathbf{k}, \Gamma)$ is a Mann pair;*
2. *for each $n \geq 1$ there is a finite $\Gamma(n) \subseteq \Gamma^n$ such that each non-degenerate solution in Γ of each equation $a_1 x_1 + \cdots + a_n x_n = 0$ with $a_1, \dots, a_n \in \mathbf{k}^\times$ lies in $\alpha \Gamma(n)$ for some $\alpha \in \Gamma$;*
3. *$\mathbf{k}^\times \cap \Gamma$ is finite, and for each $n \geq 1$ there is a finite $G(n) \subseteq G^n$ such that each non-degenerate solution in G of the equation*

$$x_1 + \cdots + x_n = 1$$

equals $(c_1 g_1, \dots, c_n g_n)$ with $c_1, \dots, c_n \in \mathbf{k}^\times$ and $(g_1, \dots, g_n) \in G(n)$.

Proof. Let $n \geq 1$. First note that if $a_1, \dots, a_n \in \mathbf{k}^\times$ and $(s_1, \dots, s_n) \in \Omega^n$, then $(s_1, \dots, s_n)$ is a non-degenerate solution of $a_1 x_1 + \cdots + a_n x_n = 1$ iff $(s_1, \dots, s_n, 1)$ is a non-degenerate solution of $a_1 x_1 + \cdots + a_n x_n - x_{n+1} = 0$.

Therefore, if (1) holds, witnessed by E for our given n as in the definition of *Mann pair* in the Introduction, then (2) holds with

$$\Gamma(n+1) := \{(\gamma_1, \dots, \gamma_n, 1) : \ (\gamma_1, \dots, \gamma_n) \in E\}.$$

Conversely, if (2) holds, take $\Gamma(n+1)$ such that $\gamma_{n+1} = 1$ for every $(\gamma_1, \dots, \gamma_n, \gamma_{n+1})$ from $\Gamma(n+1)$, and then (1) holds with

$$E := \{(\gamma_1, \dots, \gamma_n) : \ (\gamma_1, \dots, \gamma_n, 1) \in \Gamma(n+1)\}.$$

Next, observe that if (1) holds, witnessed by E for our given n, then (3) holds with $G(n) := E$. Conversely, if (3) holds, then in (3) we can take $G(n) \subseteq \Gamma^n$, and then (1) holds with

$$E := \{(\alpha_1 \gamma_1, \dots, \alpha_n \gamma_n) \in \Gamma^n : \alpha_1, \dots, \alpha_n \in \mathbf{k}^\times \cap \Gamma, (\gamma_1, \dots, \gamma_n) \in G(n)\}.$$

□

Corollary 2.2. *Suppose* $(\mathbf{k}, \Gamma)$ *is a Mann pair, and* Γ' *is a subgroup of* $\mathbf{k}^\times \Gamma$ *such that* $\mathbf{k}^\times \cap \Gamma'$ *is finite. Then* $(\mathbf{k}, \Gamma')$ *is a Mann pair.*

2.2. **Easy Mann pairs.** These are provided by the next lemma.

Lemma 2.3. *Suppose* Γ *is torsion-free. Then the following are equivalent:*

(1) *for all* $n \geq 1$ *and* $a_1, \dots, a_n \in \mathbf{k}^\times$*, the equation* $a_1 x_1 + \cdots + a_n x_n = 1$ *has no non-degenerate solution in* Γ *that is different from* $(1, \dots, 1)$*;*
(2) *whenever* $n \geq 1$ *and* $\gamma_1, \dots, \gamma_n \in \Gamma$ *are multiplicatively independent, then they are algebraically independent over* $\mathbf{k}$*.*

If these conditions are satisfied, then $(\mathbf{k}, \Gamma)$ *is a Mann pair.*

Proof. Apply Lemma 8.2 in [5] (but Γ, E, G in that lemma are $\{1\}$, $\mathbf{k}$, Γ in the present lemma). □

Since $\pi \notin \mathbb{Q}^{\mathrm{ac}}$, the group $\exp(\mathbb{Q}^{\mathrm{ac}}) \subseteq \mathbb{C}^\times$ is torsion-free, and by Lindemann's theorem on exponentials ([9], Appendix 1), condition (2) of the lemma is satisfied with $\mathbf{k} = \mathbb{Q}^{\mathrm{ac}} \subseteq \mathbb{C}$ and $\Gamma = \exp(\mathbb{Q}^{\mathrm{ac}})$. Thus $(\mathbb{Q}^{\mathrm{ac}}, \exp(\mathbb{Q}^{\mathrm{ac}}))$ is a Mann pair. Here is another application of the lemma, which applies for example to Hahn fields $K = \mathbf{k}((\Gamma))$.

Corollary 2.4. *Let* $v : K^\times \to v(K^\times)$ *be a valuation on a subfield* K *of* Ω*. Suppose* $\mathbf{k}$ *is a subfield of* K *and* Γ *a subgroup of* $K^\times$ *such that* v *is trivial on* $\mathbf{k}$ *and injective on* Γ*. Then condition* (2) *of Lemma 2.3 is satisfied, and so* $(\mathbf{k}, \Gamma)$ *is a Mann pair.*

A group with the Mann property can produce a Mann pair in an elementary extension as follows:

Corollary 2.5. *Suppose* G *is a subgroup of* $\mathbf{k}^\times$ *with the Mann property,* $(\mathbf{k}^*, G^*)$ *is an elementary extension of* $(\mathbf{k}, G)$*, and* Γ *is a subgroup of* G^* *such that* $\mathbf{k}^\times \cap \Gamma = \{1\}$*. Then* $(\mathbf{k}, \Gamma)$ *is a Mann pair.*

Proof. Note that Γ is torsion-free. Let $a_1, \dots, a_n \in \mathbf{k}^\times$, $n \geq 1$. Then the equation $a_1 x_1 + \cdots + a_n x_n = 1$ has the same finite number of non-degenerate solutions in G as in G^*, so cannot have non-degenerate solutions in Γ different from $(1, \dots, 1)$, the unique element of $G^n \cap \Gamma^n$. Hence condition (1) of Lemma 2.3 is satisfied, and so $(\mathbf{k}, \Gamma)$ is a Mann pair. □

If $\mathbf{k}$ has positive characteristic, then Lemma 2.3 is the only source of Mann pairs $(\mathbf{k}, \Gamma)$ with torsion-free Γ:

Corollary 2.6. *Suppose* $\mathbf{k}$ *has positive characteristic,* Γ *is torsion-free, and* $(\mathbf{k}, \Gamma)$ *is a Mann pair. Then conditions* (1) *and* (2) *of Lemma 2.3 are satisfied.*

Proof. Let $a_1, \dots, a_n \in \mathbf{k}^\times$, $n \geq 1$, and suppose $(\gamma_1, \dots, \gamma_n) \in \Gamma^n$ is a non-degenerate solution of $a_1 x_1 + \cdots + a_n x_n = 1$. Let ϕ be the Frobenius map on $\mathbf{k}$. Then for each m we have a non-degenerate solution

$$\big(\phi^m(\gamma_1), \dots, \phi^m(\gamma_n)\big) \in \Gamma^n$$

of the equation $\phi^m(a_1) x_1 + \cdots + \phi^m(a_n) x_n = 1$. By the equivalence (1) $\Leftrightarrow$ (2) of Lemma 2.1 this yields that all γ_i are roots of unity, so equal to 1 by torsion-freeness of Γ. □

3. Mann pairs in function fields of one variable

Let $\mathbf{k}$ be algebraically closed, and let $F \subseteq \Omega$ be a function field of one variable over $\mathbf{k}$, that is, F is a field extension of finite degree of $\mathbf{k}(t)$ for some $t \in \Omega \setminus \mathbf{k}$; in particular, F has transcendence degree 1 over $\mathbf{k}$. Below we use some standard facts about such function fields; for proofs of these facts, see Chapter I, §2 of [10].

Let $\mathcal{R}(F|\mathbf{k})$, the *Riemann space* of F over $\mathbf{k}$, be the set of all valuations $v : F^\times \to \mathbb{Z}$ on F with value group $v(F^\times) = \mathbb{Z}$ that are trivial on $\mathbf{k}$. We let v range over $\mathcal{R}(F|\mathbf{k})$. For each $f \in F^\times$ we have $v(f) \neq 0$ for only finitely many v, and $\sum_v v(f) = 0$. Let $\mathcal{D}(F|\mathbf{k})$ be the group of *divisors* of F over $\mathbf{k}$, that is,

$$\mathcal{D}(F|\mathbf{k}) \ := \ \bigoplus_v \mathbb{Z}v$$

is the free abelian group on the Riemann space of F over $\mathbf{k}$. To $f \in F^\times$ we assign its *principal divisor* $(f) := \sum_v v(f)v \in \mathcal{D}(F|\mathbf{k})$. The group morphism

$$F^\times \to \mathcal{D}(F|\mathbf{k}), \quad f \mapsto (f)$$

has kernel $\mathbf{k}^\times$. In particular, if Γ is a subgroup of $F^\times$ with $\mathbf{k}^\times \cap \Gamma = \{1\}$, then this morphism is injective on Γ, and so the image of Γ under this morphism is an isomorphic copy of Γ inside the free abelian group $\mathcal{D}(F|\mathbf{k})$. It follows that each such Γ is free as an abelian group.

Given finite $S \subseteq \mathcal{R}(F|\mathbf{k})$, an *$S$-unit* is an element $u \in F^\times$ such that $v(u) = 0$ for all $v \notin S$.

Lemma 3.1. *Let L be a finite-dimensional $\mathbf{k}$-linear subspace of F and let $\Gamma \subseteq F^\times$ be finitely generated with $\mathbf{k}^\times \cap \Gamma = \{1\}$. Then $L \cap \Gamma$ is finite.*

Proof. Let $b_1, \dots, b_m$ be a basis of the $\mathbf{k}$-linear space L, and let $\gamma_1, \dots, \gamma_n$ generate the group Γ. Take a finite $S \subseteq \mathcal{R}(F|\mathbf{k})$ such that all b_i and γ_j are S-units. Take a natural number d such that $v(b_i) \geq -d$ for all $v \in S$ and $i = 1, \dots, m$. Then $v(f) \geq 0$ for all $f \in L$ and v outside S, and $v(f) \geq -d$ for all $f \in L$ and $v \in S$.

Suppose now that $\gamma \in L \cap \Gamma$. Then $v(\gamma) = 0$ for v outside S and $v(\gamma) \geq -d$ for $v \in S$. In view of $\sum_v v(\gamma) = 0$, this gives $v(\gamma) \leq |S|d$ for all $v \in S$. It follows that the image of $L \cap \Gamma$ in $\mathcal{D}(F|\mathbf{k})$ is finite, and thus $L \cap \Gamma$ is finite. □

Let $u_1, \dots, u_n \in F$ not be all zero. We define their *height* by

$$H(u_1, \dots, u_n) := -\sum_v \min\{v(u_1), \dots, v(u_n)\}.$$

This height is projective: $H(fu_1, \dots, fu_n) = H(u_1, \dots, u_n)$ for $f \in F^\times$.

Example. Let $F = \mathbf{k}(t)$ with t transcendental over $\mathbf{k}$. Suppose that the polynomials $u_1, \dots, u_n \in \mathbf{k}[t]$ have no common zero in $\mathbf{k}$, $n \geq 1$. It is easy to check that then $H(u_1, \dots, u_n) = \max\{\deg_t u_1, \dots, \deg_t u_n\}$.

The following important bound is from [3]:

Let Ω have characteristic zero, let g be the genus of the function field $F|\mathbf{k}$, and let S be a finite subset of $\mathcal{R}(F|\mathbf{k})$ and $n \geq 2$. Suppose $u_1, \dots, u_n$ are S-units and $(u_1, \dots, u_n)$ is a non-degenerate solution of $x_1 + \cdots + x_n = 0$. Then

$$H(u_1, \dots, u_n) \leq \frac{1}{2}(n-1)(n-2)\{|S| + \max(0, 2\,\mathrm{g} - 2)\}.$$

In combination with the previous lemma this has the following consequence:

Corollary 3.2. *Suppose* Ω *has characteristic zero and* $\Gamma \subseteq F^\times$ *is finitely generated with* $\mathbf{k}^\times \cap \Gamma = \{1\}$. *Then* $(\mathbf{k}, \Gamma)$ *is a Mann pair.*

Proof. Take a finite $S \subseteq \mathcal{R}(F|\mathbf{k})$ such that all $\gamma \in \Gamma$ are S-units. Let $n \geq 2$ and let $a_1, \dots, a_n \in \mathbf{k}^\times$ and $\gamma_1, \dots, \gamma_n \in \Gamma$ be such that $(\gamma_1, \dots, \gamma_n)$ is a non-degenerate solution of $a_1x_1 + \cdots + a_nx_n = 0$. Dividing by γ_n we arrange $\gamma_n = 1$, and we need to show that this leaves only finitely many possibilities for $(\gamma_1, \dots, \gamma_{n-1})$. Now by the bound above we have $v(\gamma_i) \geq -N$ for all $v \in S$ and $i = 1, \dots, n-1$, where $N := \frac{1}{2}(n-1)(n-2)\{|S| + \max(0, 2\,\mathrm{g} - 2)\}$, so each γ_i lies in the $\mathbf{k}$-linear subspace

$$L := \{f \in F : \ v(f) \geq 0 \text{ for all } v \notin S,\ v(f) \geq -N \text{ for all } v \in S\}$$

of F. Now L is finite-dimensional by [10], p. 7. In view of Lemma 3.1 this gives the desired finiteness. □

Example. To illustrate the effective nature of this proof, assume that Ω has characteristic zero, and consider the case $F = \mathbf{k}(t)$ of a *rational* function field (so $\mathrm{g} = 0$), where Γ is generated as a group by

$$t - c_1, \dots, t - c_M, \qquad \text{with distinct } c_1, \dots, c_M \in \mathbf{k}.$$

Let $n \geq 2$ be given. Call a tuple $(\gamma_1, \dots, \gamma_n) \in \Gamma^n$ *reduced* if all γ_i lie in $\mathbf{k}[t]$ and $\gamma_1, \dots, \gamma_n$ have no common zero in $\mathbf{k}$ when viewed as polynomials in t over $\mathbf{k}$. Let $a_1, \dots, a_n \in \mathbf{k}^\times$. Any non-degenerate solution in Γ of the equation

$$a_1x_1 + \cdots + a_nx_n = 0$$

can be multiplied by an element of Γ to give a non-degenerate reduced solution $(\gamma_1, \dots, \gamma_n)$. In this situation the inequality from [3] yields

$$\max\{\deg_t \gamma_1, \dots, \deg_t \gamma_n\} \leq \frac{1}{2}(n-1)(n-2)(M+1),$$

which is satisfied by only finitely many reduced tuples $(\gamma_1, \dots, \gamma_n)$, which we can list explicitly. Given a reduced tuple $(\gamma_1, \dots, \gamma_n)$ satisfying the inequality, the existence of $a_1, \dots, a_n \in \mathbf{k}^\times$ such that $(\gamma_1, \dots, \gamma_n)$ is a non-degenerate solution of $a_1x_1 + \cdots + a_nx_n = 0$ is equivalent (effectively) to the existence of a solution to a certain finite system of linear equations and inequations with coefficients in the field $\mathbb{Q}(c_1, \dots, c_M)$.

4. Allowing coefficients from Ω

In this section we consider Γ as acting on itself by multiplication, making Γ into a Γ-set as defined in [5], §4. In particular, any subset of Γ^n definable in the Γ-set Γ is definable in the group Γ. Our aim is to prove the following result and some related facts.

Proposition 4.1. *Let* $(\mathbf{k}, \Gamma)$ *be a Mann pair and* $r_1, \dots, r_n \in \Omega$. *Then*

$$\{(x, y) \in \mathbf{k}^n \times \Gamma^n : \ r_1x_1y_1 + \cdots + r_nx_ny_n = 0\}$$

is a finite union of sets $X \times Y$ *with* X *a* $\mathbf{k}$*-linear subspace of* $\mathbf{k}^n$ *and* $Y \subseteq \Gamma^n$ *defined in the* Γ*-set* Γ *by a conjunction of atoms in the language of* Γ*-sets.*

We first show this for $r_1, \dots, r_n \in \mathbf{k}$, next for $r_1, \dots, r_n \in \mathbf{k}(\Gamma)$, and then in general.

4.1. **Notations.** In this section we let $\vec{a} = (a_1, \ldots, a_n)$ and $\vec{b} = (b_1, \ldots, b_n)$ range over $\mathbf{k}^n$, and α, β, γ over Γ, and

$$\vec{\alpha} = (\alpha_1, \ldots, \alpha_n), \quad \vec{\beta} = (\beta_1, \ldots, \beta_n), \quad \vec{\gamma} = (\gamma_1, \ldots, \gamma_n)$$

over Γ^n. Let $n \geq 1$ and put

$$\Sigma_n(\mathbf{k}, \Gamma) : = \{(\vec{a}, \vec{\gamma}) \in \mathbf{k}^n \times \Gamma^n : \ a_1\gamma_1 + \cdots + a_n\gamma_n = 0\},$$
$$\Sigma_n(\mathbf{k}, \Gamma; \vec{\gamma}) : = \{\vec{a} \in \mathbf{k}^n : \ (\vec{a}, \vec{\gamma}) \in \Sigma_n(\mathbf{k}, \Gamma)\}.$$

Imposing non-degeneracy yields the set $\Sigma_n^{\mathrm{nd}}(\mathbf{k}, \Gamma)$ of all $(\vec{a}, \vec{\gamma}) \in (\mathbf{k}^\times)^n \times \Gamma^n$ such that $\vec{\gamma}$ is a non-degenerate solution of $a_1x_1 + \cdots + a_nx_n = 0$. We also introduce for each $\vec{\gamma}$ the corresponding section

$$\Sigma_n^{\mathrm{nd}}(\mathbf{k}, \Gamma; \vec{\gamma}) := \{\vec{a} \in (\mathbf{k}^\times)^n : \ (\vec{a}, \vec{\gamma}) \in \Sigma_n^{\mathrm{nd}}(\mathbf{k}, \Gamma)\}.$$

If $(\vec{a}, \vec{\gamma}) \in \Sigma_n^{\mathrm{nd}}(\mathbf{k}, \Gamma)$, then $(\vec{a}, \alpha\vec{\gamma}) \in \Sigma_n^{\mathrm{nd}}(\mathbf{k}, \Gamma)$, so $\Sigma_n^{\mathrm{nd}}(\mathbf{k}, \Gamma)$ is a union of sets of the form $\Sigma_n^{\mathrm{nd}}(\mathbf{k}, \Gamma; \vec{\gamma}) \times \Gamma\vec{\gamma}$. Being a Mann pair can now be expressed as follows:

$(\mathbf{k}, \Gamma)$ *is a Mann pair iff for each* $n \geq 1$ *there is a finite* $\Gamma(n) \subseteq \Gamma^n$ *such that*

$$\Sigma_n^{\mathrm{nd}}(\mathbf{k}, \Gamma) = \bigcup_{\vec{\gamma} \in \Gamma(n)} \Sigma_n^{\mathrm{nd}}(\mathbf{k}, \Gamma; \vec{\gamma}) \times \Gamma\vec{\gamma}.$$

Whenever $(\mathbf{k}, \Gamma)$ is a Mann pair we let $\Gamma(n)$ for $n \geq 1$ be as above.

We also want to allow coefficients outside $\mathbf{k}$ and accordingly, given $\vec{r} = (r_1, \ldots, r_n) \in \Omega^n$, we let $\Sigma(\vec{r}, \mathbf{k}, \Gamma)$ be the set of all

$$(\vec{a}, \vec{\gamma}) = (a_1, \ldots, a_n, \gamma_1, \ldots, \gamma_n) \in \mathbf{k}^n \times \Gamma^n$$

such that $\vec{\gamma}$ is a solution of $r_1a_1x_1 + \cdots + r_na_nx_n = 0$, and let $\Sigma^{\mathrm{nd}}(\vec{r}, \mathbf{k}, \Gamma)$ be the set of all $(\vec{a}, \vec{\gamma}) \in (\mathbf{k}^\times)^n \times \Gamma^n$ such that $\vec{\gamma}$ is a non-degenerate solution of $r_1a_1x_1 + \cdots + r_na_nx_n = 0$. In particular, $\Sigma^{\mathrm{nd}}(\vec{r}, \mathbf{k}, \Gamma) = \emptyset$ if $n \geq 2$ and $r_i = 0$ for some $i \in \{1, \ldots, n\}$. Also, let $\Sigma^{\mathrm{nd}}(\vec{r}, \mathbf{k}, \Gamma; \vec{\gamma})$ be the set of $\vec{a} \in (\mathbf{k}^\times)^n$ such that $(\vec{a}, \vec{\gamma}) \in \Sigma^{\mathrm{nd}}(\vec{r}, \mathbf{k}, \Gamma)$.

4.2. **Mann pairs over F.** Besides $\mathbf{k}$ we also let F denote a subfield of Ω. Usually, $\mathbf{k}$ serves as basefield, and F will be an extension of $\mathbf{k}$.

We say that $(\mathbf{k}, \Gamma)$ is a *Mann pair over* F if for every tuple $\vec{r} = (r_1, \ldots, r_n)$ from $F^\times$, $n \geq 2$, there is a finite subset $\Gamma(\vec{r})$ of Γ^n, such that

$$\Sigma^{\mathrm{nd}}(\vec{r}, \mathbf{k}, \Gamma) = \bigcup_{\vec{\gamma} \in \Gamma(\vec{r})} \Sigma^{\mathrm{nd}}(\vec{r}, \mathbf{k}, \Gamma; \vec{\gamma}) \times \Gamma\vec{\gamma}.$$

It is clear that $(\mathbf{k}, \Gamma)$ is a Mann pair iff it is a Mann pair over $\mathbf{k}$. We proceed to show that if $(\mathbf{k}, \Gamma)$ is a Mann pair, then it is a Mann pair over Ω.

Let $\vec{r} = (r_1, \ldots, r_n) \in F^n$, $n \geq 1$. Let $\mathcal{P}$ be a partition of $\{1, \ldots, n\}$ into distinct sets. Then we define $\Sigma_{\mathcal{P}}(\vec{r}, \mathbf{k}, \Gamma)$ to be the set of all

$$(a_1, \ldots, a_n, \beta_1, \ldots, \beta_n) \in \mathbf{k}^n \times \Gamma^n$$

such that for $I \in \mathcal{P}$ the tuple $(\beta_i)_{i \in I}$ is a non-degenerate solution of

$$\sum_{i \in I} r_ia_ix_i = 0$$

With $\vec{r}_I := (r_i)_{i\in I}$ for $I \in \mathcal{P}$, this means

$$\Sigma_{\mathcal{P}}(\vec{r},\mathbf{k},\Gamma) \;=\; \prod_{I\in\mathcal{P}} \Sigma^{\mathrm{nd}}(\vec{r}_I,\mathbf{k},\Gamma).$$

Suppose now that $(\mathbf{k},\Gamma)$ is an Mann pair over F. Then we have for $I \in \mathcal{P}$ a finite $\Gamma(\vec{r}_I) \subseteq \Gamma^I$ such that

$$\Sigma^{\mathrm{nd}}(\vec{r}_I,\mathbf{k},\Gamma) \;=\; \bigcup_{\vec{\gamma}\in\Gamma(\vec{r}_I)} \Sigma^{\mathrm{nd}}(\vec{r}_I,\mathbf{k},\Gamma;\vec{\gamma}) \times \Gamma\vec{\gamma}.$$

It follows that

$$\Sigma_{\mathcal{P}}(\vec{r},\mathbf{k},\Gamma) \;=\; \prod_{I\in\mathcal{P}} \bigcup_{\vec{\gamma}\in\Gamma(\vec{r}_I)} \Sigma^{\mathrm{nd}}(\vec{r}_I,\mathbf{k},\Gamma;\vec{\gamma}) \times \Gamma\vec{\gamma}.$$

Thus, with $\vec{\gamma}_I := (\gamma_i)_{i\in I}$ and $\Gamma(\vec{r}) = \Gamma(\vec{r},\mathcal{P}) := \prod_{I\in\mathcal{P}} \Gamma(\vec{r}_I) \subseteq \Gamma^n$,

$$\begin{aligned}\Sigma_{\mathcal{P}}(\vec{r},\mathbf{k},\Gamma) \;&=\; \bigcup_{\vec{\gamma}\in\Gamma(\vec{r})} \prod_{I\in\mathcal{P}} \big[\Sigma^{\mathrm{nd}}\big(\vec{r}_I,\mathbf{k},\Gamma;\vec{\gamma}_I\big) \times \Gamma\vec{\gamma}_I\big] \\ &\subseteq\; \bigcup_{\vec{\gamma}\in\Gamma(\vec{r})} \prod_{I\in\mathcal{P}} \big[\Sigma\big(\vec{r}_I,\mathbf{k},\Gamma;\vec{\gamma}_I\big) \times \Gamma\vec{\gamma}_I\big].\end{aligned}$$

The last product set is contained in $\Sigma(\vec{r},\mathbf{k},\Gamma)$ under an obvious identification, so with $\mathcal{P}$ ranging over the partitions of $\{1,\dots,n\}$, we get

$$\Sigma(\vec{r},\mathbf{k},\Gamma) \;=\; \bigcup_{\mathcal{P}} \Sigma_{\mathcal{P}}(\vec{r},\mathbf{k},\Gamma) \;=\; \bigcup_{\mathcal{P}} \bigcup_{\vec{\gamma}\in\Gamma(\vec{r},\mathcal{P})} \prod_{I\in\mathcal{P}} \big[\Sigma\big(\vec{r}_I,\mathbf{k},\Gamma;\vec{\gamma}_I\big) \times \Gamma\vec{\gamma}_I\big].$$

This yields the following result.

Lemma 4.2. *Suppose $(\mathbf{k},\Gamma)$ is a Mann pair over F, and $\vec{r} \in F^n$, $n \geq 1$. Then $\Sigma(\vec{r},\mathbf{k},\Gamma)$ is a finite union of sets $P \times Q$, where P is a $\mathbf{k}$-linear subspace of $\mathbf{k}^n$, and $Q \subseteq \Gamma^n$ is defined in Γ by a conjunction of atoms in the language of Γ-sets.*

Note that the intersection of two sets of the form $P \times Q$, where P, Q are as in Lemma 4.2, is again of the same form. Hence, if $(\mathbf{k},\Gamma)$ is a Mann pair over F, and $\vec{r},\vec{s} \in F^n$, $n \geq 1$, then $\Sigma(\vec{r},\mathbf{k},\Gamma) \cap \Sigma(\vec{s},\mathbf{k},\Gamma)$ is a finite union of sets $P \times Q$, where P, Q are as in Lemma 4.2. Next we prove a converse of Lemma 4.2.

Lemma 4.3. *Let $\vec{r} \in \Omega^n$. Suppose that*

$$\Sigma(\vec{r},\mathbf{k},\Gamma) \;=\; \bigcup_{j=1}^{k} P_j \times Q_j,$$

where $P_1,\dots,P_k$ are subsets of $\mathbf{k}^n$, and each Q_j is a subset of Γ^n defined in Γ by a conjunction of atoms in the language of Γ-sets. Then there is a finite subset $\Gamma(\vec{r})$ of Γ^n such that

$$\Sigma^{\mathrm{nd}}(\vec{r},\mathbf{k},\Gamma) \;=\; \bigcup_{\vec{\gamma}\in\Gamma(\vec{r})} \Sigma^{\mathrm{nd}}(\vec{r},\mathbf{k},\Gamma;\vec{\gamma}) \times \Gamma\vec{\gamma}.$$

Proof. We can assume $\Gamma \neq \{1\}$. Let $j \in \{1,\dots,k\}$ be such that $Q_j \neq \emptyset$. By the discussion in [5] just after Corollary 4.2, we can take $\vec{\gamma} \in \Gamma^n$ and a partition of $\{1,\dots,n\}$ into distinct sets $I(1),\dots,I(l)$ such that

$$Q_j \;=\; \Gamma\vec{\gamma}(1) \times \cdots \times \Gamma\vec{\gamma}(l) \;\subseteq\; \prod_{\lambda=1}^{l} \Gamma^{I(\lambda)} = \Gamma^n,$$

where $\vec{\gamma}(\lambda) := \{(\gamma_i)\}_{i\in I(\lambda)} \in \Gamma^{I(\lambda)}$ for $\lambda = 1,\dots,l$.

Claim. Let $l > 1$, $\vec{a} \in P_j$ and $\vec{\beta} \in Q_j$. Then $(a_1\beta_1,\dots,a_n\beta_n)$ is a degenerate solution of $r_1x_1 + \cdots + r_nx_n = 0$.

Proof of the claim. For each $\alpha \in \Gamma$,

$$\sum_{i\in I(1)} r_ia_i\beta_i + \sum_{i\notin I(1)} r_ia_i\beta_i = 0 = \sum_{i\in I(1)} r_ia_i\beta_i + \sum_{i\notin I(1)} r_ia_i\alpha\beta_i.$$

So $\sum_{i\notin I(1)} r_ia_i\beta_i = \alpha\sum_{i\notin I(1)} r_ia_i\beta_i$ for all $\alpha \in \Gamma$. Hence $\sum_{i\notin I(1)} r_ia_i\beta_i = 0$, and thus $(a_1\beta_1,\dots,a_n\beta_n)$ is a degenerate solution, proving the claim.

As a result of this claim, we get a subset J of $\{1,\dots,k\}$, and a tuple $\vec{\gamma}_j \in \Gamma^n$ for each $j \in J$ such that

$$\Sigma^{\mathrm{nd}}(\vec{r},\mathbf{k},\Gamma) \ \subseteq\ \bigcup_{j\in J} P_j \times \Gamma\vec{\gamma}_j.$$

Since obviously $\bigcup_{j\in J}\Sigma^{\mathrm{nd}}(\vec{r},\mathbf{k},\Gamma;\vec{\gamma}_j)\times\Gamma\vec{\gamma}_j \subseteq \Sigma^{\mathrm{nd}}(\vec{r},\mathbf{k},\Gamma)$, this yields

$$\Sigma^{\mathrm{nd}}(\vec{r},\mathbf{k},\Gamma) \ =\ \bigcup_{j\in J}\Sigma^{\mathrm{nd}}(\vec{r},\mathbf{k},\Gamma;\vec{\gamma}_j)\times\Gamma\vec{\gamma}_j.$$

□

Note: $P_1,\dots,P_k$ in this lemma are not assumed to be linear subspaces of $\mathbf{k}^n$. Applying the lemma with $\vec{r} = (1,\dots,1)$, we obtain

Lemma 4.4. *Suppose for each $n \geq 1$ the set $\Sigma_n(\mathbf{k},\Gamma)$ is a finite union of sets $P \times Q$ with $P \subseteq \mathbf{k}^n$ and Q a subset of Γ^n defined in Γ by a positive quantifier-free formula in the language of Γ-sets. Then $(\mathbf{k},\Gamma)$ is a Mann pair.*

Next we improve Lemma 4.4 and obtain a strong version of the implication (2) $\Longrightarrow$ (1) of Theorem 1.2. We shall use facts from [5], but alert the reader that in the statement and proof of Proposition 5.11 of [5], "subgroups of G^n" should be "cosets of subgroups of G^n", and "B_n" should be "B_m".

Proposition 4.5. *Suppose for each $n \geq 1$ the set $\Sigma_n(\mathbf{k},\Gamma)$ is a finite union of sets $X \times Y$ where $X \subseteq \mathbf{k}^n$ and where $Y \subseteq \Gamma^n$ is a boolean combination of cosets of subgroups of Γ^n. Then $(\mathbf{k},\Gamma)$ is a Mann pair.*

Proof. Let $n \geq 1$. For $(x,y) \in \mathbf{k}^n \times \Gamma^n$ we have

$$x_1y_1 + \cdots + x_ny_n = 1 \iff \big((x_1,\dots,x_n,-1),(y_1,\dots,y_n,1)\big) \in \Sigma_{n+1}(\mathbf{k},\Gamma).$$

Using the hypothesis with $n+1$ instead of n it follows easily that the set

$$\{(x,y) \in \mathbf{k}^n \times \Gamma^n :\ x_1y_1 + \cdots + x_ny_n = 1\}$$

is a finite union of sets $X \times Y$ where $X \subseteq \mathbf{k}^n$ and where $Y \subseteq \Gamma^n$ is a boolean combination of cosets of subgroups of Γ^n. Then Proposition 5.11 of [5] yields that Γ has the Mann property. Next,

$$\Sigma_n(\mathbf{k},\Gamma) = \bigcup_{i\in I} X_i \times Y_i$$

with finite I and each $X_i \subseteq \mathbf{k}^n$, and where each $Y_i \subseteq \Gamma^n$ is a boolean combination of cosets of subgroups of Γ^n. We can also arrange that $X_i \neq \emptyset$ for all i and $X_i \cap X_j = \emptyset$ for all distinct $i,j \in I$.

By Corollary 5.1 and Lemma 5.5 in [5], it follows that each set Y_i is definable in Γ by a positive quantifier-free formula in the language of Γ-sets. Then Lemma 4.4 yields that $(\mathbf{k},\Gamma)$ is a Mann pair. □

Lemma 4.6. *If $(\mathbf{k},\Gamma)$ is a Mann pair, then it is a Mann pair over $\mathbf{k}(\Gamma)$.*

Proof. Assume $(\mathbf{k},\Gamma)$ is a Mann pair, and let $r_1,\dots,r_n \in \mathbf{k}(\Gamma)^\times$. By Lemma 4.3, it suffices to prove that $\Sigma(\vec{r},\mathbf{k},\Gamma)$ is a finite union of sets $P\times Q$, where $P\subseteq \mathbf{k}^n$, and $Q\subseteq \Gamma^n$ is defined by a finite conjunction of atoms in the language of Γ-sets. We may assume that $r_1,\dots,r_n\in\mathbf{k}[\Gamma]$. Then $r_i=\sum_{j=1}^k a_{ij}\beta_j$ with $a_{ij}\in\mathbf{k}$, and $\beta_j\in\Gamma$ for $i=1,\dots,n$ and $j=1,\dots,k$.

Let $\vec{a}=(a_{11},\dots,a_{1k},\dots,a_{n1},\dots,a_{nk})\in\mathbf{k}^{nk}$. Then

$$(c_1,\dots,c_n,\gamma_1,\dots,\gamma_n)\in\Sigma(\vec{r},\mathbf{k},\Gamma)$$
$$\Longleftrightarrow$$
$$(c_1,\dots,c_1,\dots,c_n,\dots,c_n,\beta_1\gamma_1,\dots,\beta_k\gamma_1,\dots,\beta_1\gamma_n,\dots,\beta_k\gamma_n)\in\Sigma(\vec{a},\mathbf{k},\Gamma).$$

By Lemma 4.2, $\Sigma(\vec{a},\mathbf{k},\Gamma)$ is a finite union of sets $P\times Q$, where $P\subseteq\mathbf{k}^{nk}$, and $Q\subseteq\Gamma^{nk}$ is defined in Γ by a conjunction of atoms in the language of Γ-sets. Fix such P,Q, and define

$$P' := \{(c_1,\dots,c_n)\in\mathbf{k}^n:\ (c_1,\dots,c_1,\dots,c_n,\dots,c_n)\in P\},$$
$$Q' := \{(\gamma_1,\dots,\gamma_n)\in\Gamma^n:\ (\beta_1\gamma_1,\dots,\beta_k\gamma_1,\dots,\beta_1\gamma_n,\dots,\beta_k\gamma_n)\in Q\}.$$

Then for $(c_1,\dots,c_n,\gamma_1,\dots,\gamma_n)\in\mathbf{k}^n\times\Gamma^n$,

$$(c_1,\dots,c_n,\gamma_1,\dots,\gamma_n)\in P'\times Q'$$
$$\Longleftrightarrow$$
$$(c_1,\dots,c_1,\dots,c_n,\dots,c_n,\beta_1\gamma_1,\dots,\beta_k\gamma_n)\in P\times Q.$$

Hence, $\Sigma(\vec{a},\mathbf{k},\Gamma)$ being a finite union of sets $P\times Q$ with P and Q as above, $\Sigma(\vec{r},\mathbf{k},\Gamma)$ is the union of the corresponding sets $P'\times Q'$. This finishes the proof of the lemma. □

Proposition 4.7. *If $(\mathbf{k},\Gamma)$ is a Mann pair, then it is a Mann pair over Ω.*

Proof. Let $\vec{r}=(r_1,\dots,r_n)\in\Omega^n$. Take a basis $b_1,\dots,b_m$ of the $\mathbf{k}(\Gamma)$-linear space $\mathbf{k}(\Gamma)r_1+\cdots+\mathbf{k}(\Gamma)r_n$. Then $r_j=\sum_{i=1}^m r_{ij}b_i$ with $r_{ij}\in\mathbf{k}(\Gamma)$, so that for all $(\vec{c},\vec{\gamma})\in\mathbf{k}^n\times\Gamma^n$ we have: $(\vec{c},\vec{\gamma})\in\Sigma(\vec{r},\mathbf{k},\Gamma)$ if and only if $(c_1\gamma_1,\dots,c_n\gamma_n)$ is a solution of the system

$$\begin{aligned} r_{11}x_1+\cdots+r_{1n}x_n&=0\\ r_{21}x_1+\cdots+r_{2n}x_n&=0\\ &\ \vdots\\ r_{m1}x_1+\cdots+r_{mn}x_n&=0. \end{aligned}$$

Suppose now that $(\mathbf{k},\Gamma)$ is a Mann pair. Then it is a Mann pair over $\mathbf{k}(\Gamma)$ by the previous lemma. It remains to use Lemmas 4.2 and 4.3. □

Clearly, this proposition and Lemma 4.2 yield Proposition 4.1.

For any n-tuple $k=(k_1,\dots,k_n)\in\mathbb{Z}^n$, consider the *character*

$$\chi_k:\left(\Omega^\times\right)^n\to\Omega^\times,\quad \chi_k(y_1,\dots,y_n):=y_1^{k_1}\cdots y_n^{k_n}.$$

This is a multiplicative group homomorphism. For any $e \in \mathbb{N}$, let $\mathcal{D}(n,e)$ be the finite collection of subgroups of $(\Omega^\times)^n$ that are intersections of kernels of characters χ_k with $|k| = |k_1| + \cdots + |k_n| \le e$.

Proposition 4.8. *Let $(\mathbf{k},\Gamma)$ be a Mann pair, let $X_1,\dots,X_m,Y_1,\dots,Y_n$ be distinct indeterminates, and let the polynomials $f_1,\dots,f_M \in \Omega[X,Y]$ have degree $\le d$ in $X=(X_1,\dots,X_m)$ and degree $\le e$ in $Y=(Y_1,\dots,Y_n)$, where $M,d,e \in \mathbb{N}$, and put*

$$Z := \{(x,y) \in \mathbf{k}^m \times \Gamma^n : f_1(x,y) = \cdots = f_M(x,y) = 0\}.$$

Then Z is a finite union of sets $P \times Q$, where

$$P = \{x \in \mathbf{k}^m : g_1(x) = \cdots = g_N(x) = 0\}$$

for suitable $N \in \mathbb{N}$ and polynomials $g_1,\dots,g_N \in \mathbf{k}[X]$ of degree $\le d$, and where $Q \subseteq \Gamma^n$ is a coset of a subgroup $D \cap \Gamma^n$ of Γ^n with $D \in \mathcal{D}(n,e)$.

Proof. The intersection of finitely many cosets of such subgroups is either empty or again a coset of such a subgroup. Hence we may (and shall) assume that $M=1$. Put $f := f_1$. Then $f = \sum_{(i,j)\in I\times J} a_{ij}X^iY^j$ where all $a_{ij} \in \Omega$ and I is the set of multi-indices $i=(i_1,\dots,i_m) \in \mathbb{N}^m$ with $|i| = i_1+\cdots+i_m \le d$ and J is the set of multi-indices $j=(j_1,\dots,j_n) \in \mathbb{N}^n$ with $|j| = j_1+\cdots+j_n \le e$. By Proposition 4.7 and Lemma 4.2 the set

$$\{\big((x_i)_{i\in I},(y_j)_{j\in J}\big) \in \mathbf{k}^I \times \Gamma^J : \sum_{(i,j)\in I\times J} a_{ij}x_iy_j = 0\}$$

is a finite union of subsets of $\mathbf{k}^I \times \Gamma^J$ of the form

$$V \times \{y \in \Gamma^J :\ \gamma_1 y_{i(1)} = y_{j(1)},\dots,\gamma_k y_{i(k)} = y_{j(k)}\}$$

with V a $\mathbf{k}$-linear subspace of $\mathbf{k}^I$, $\gamma_1,\dots,\gamma_k \in \Gamma$, and with $k \in \mathbb{N}$ and indices $i(1), j(1)\dots, i(k), j(k)$ in J. It remains to observe that for such $\gamma_1,\dots,\gamma_k$ and $i(1), j(1),\dots, i(k), j(k)$ the set

$$\{y \in \Gamma^J :\ \gamma_1\chi_{i(1)}(y) = \chi_{j(1)}(y),\dots,\gamma_k\chi_{i(k)}(y) = \chi_{j(k)}(y)\}$$

is a coset of the subgroup $D \cap \Gamma^n$ of Γ^n where D is the intersection of the kernels of $\chi_{i(1)-j(1)},\dots,\chi_{i(k)-j(k)}$. ☐

5. Robustness of Mann pairs

In this section we assume that $(\mathbf{k},\Gamma)$ is a Mann pair. In addition, $K \supseteq \mathbf{k}$ is a subfield of Ω, and Δ, Γ' are subgroups of $\Omega^\times$ with $\Gamma' \supseteq \Gamma$. We claim:

1. $[K:\mathbf{k}] < \infty \ \Rightarrow\ (K,\Gamma)$ is a Mann pair;
2. $[\Gamma':\Gamma] < \infty \ \Rightarrow\ (\mathbf{k},\Gamma')$ is a Mann pair;
3. $\big(K \supseteq \mathbf{k}(\Gamma)$ and (K,Δ) is a Mann pair$\big) \ \Rightarrow\ (\mathbf{k},\Gamma\Delta)$ is a Mann pair;
4. $\big(K$ is linearly disjoint from $\mathbf{k}(\Gamma)$ over $\mathbf{k}\big) \Rightarrow\ (K,\Gamma)$ is a Mann pair;

Proof of (1)**.** Here K is an extension field of finite degree m over $\mathbf{k}$. Let $b_1,\dots,b_m$ be a basis of the $\mathbf{k}$-linear space K. Let $n \ge 1$, and consider $(\vec{a},\vec{\gamma}) \in K^n \times \Gamma^n$. Then with $a_j = \sum_{i=1}^m a_{ij}b_i$ $(1 \le j \le n,$ all $a_{ij} \in \mathbf{k})$,

$$\begin{aligned}(\vec{a},\vec{\gamma}) \in \Sigma_n(K,\Gamma) &\iff \sum_{j=1}^n (a_{1j}b_1 + \cdots + a_{mj}b_m)\gamma_j = 0 \\ &\iff F(a_{11},\ldots,a_{mn},\gamma_1,\ldots,\gamma_n) = 0\end{aligned}$$

$$\text{with } F := \sum_{j=1}^n (X_{1j}b_1 + \cdots + X_{mj}b_m)Y_j \in K[X_{11},\ldots,X_{mn},Y_1,\ldots,Y_n].$$

By Proposition 4.8, the set

$$\{\big((c_{ij}),\gamma\big) \in \mathbf{k}^{mn} \times \Gamma^n : F(c_{11},\ldots,c_{mn},\gamma_1,\ldots,\gamma_n) = 0\}$$

is a union $\bigcup_{i\in I} P_i \times Q_i$ with finite I, where each $P_i \subseteq \mathbf{k}^{mn}$, and each $Q_i \subseteq \Gamma^n$ is a coset of a subgroup of Γ^n. Therefore we have $\Sigma_n(K,\Gamma) = \bigcup_{i\in I} P_i' \times Q_i$ with $P_i' = \{\vec{a} \in K^n : (a_{ij}) \in P_i\}$, where $\vec{a}$ determines (a_{ij}) as above.

Corollary 5.1. *If $\gamma \in \Gamma$ is algebraic over $\mathbf{k}$, then γ is a root of unity. In particular, if $(\mathbb{Q},\Gamma)$ is a Mann pair with $\Gamma \subseteq A^\times$, then $\Gamma \subseteq \mathbb{U}$.*

Proof of (2)**.** Assume $\Gamma' = \bigcup_{\alpha\in C} \alpha\Gamma$ where $C \subseteq \Gamma'$ is finite. Then

$$\Sigma_n(\mathbf{k},\Gamma') = \bigcup_{\vec{\alpha}\in C^n} \{(\vec{a},\vec{\alpha}\vec{\gamma}) :\ (\vec{a},\vec{\gamma}) \in \Sigma(\vec{\alpha},\mathbf{k},\Gamma),$$

and it remains to use Proposition 4.7, Lemma 4.2 and Proposition 4.5.

Proof of (3)**.** Assume $K \supseteq \mathbf{k}(\Gamma)$ and (K,Δ) is a Mann pair. Let $n \ge 1$ and take finite $\Delta(n) \subseteq \Delta^n$ such that

$$\Sigma_n^{\mathrm{nd}}(K,\Delta) = \bigcup_{\vec{\delta}\in\Delta(n)} \Sigma_n^{\mathrm{nd}}(K,\Delta;\vec{\delta}) \times \Delta\vec{\delta}.$$

For each $\vec{\delta} \in \Delta(n)$ we have a finite $\Gamma(\vec{\delta}) \subseteq \Gamma^n$ such that

$$\Sigma^{\mathrm{nd}}(\vec{\delta},\mathbf{k},\Gamma) = \bigcup_{\vec{\gamma}\in\Gamma(\vec{\delta})} \Sigma^{\mathrm{nd}}(\vec{\delta},\mathbf{k},\Gamma;\vec{\gamma}) \times \Gamma\vec{\gamma}.$$

It follows that, with $\vec{\gamma}\vec{\delta} := (\gamma_1\delta_1,\ldots,\gamma_n\delta_n)$, we have

$$\Sigma_n^{\mathrm{nd}}(\mathbf{k},\Gamma\Delta) = \bigcup_{\vec{\delta}\in\Delta(n)} \bigcup_{\vec{\gamma}\in\Gamma(\vec{\delta})} \Sigma_n^{\mathrm{nd}}(\mathbf{k},\Gamma\Delta;\vec{\gamma}\vec{\delta}) \times \Gamma\Delta\vec{\gamma}\vec{\delta}.$$

Proof of (4)**.** Assume K is linearly disjoint from $\mathbf{k}(\Gamma)$ over $\mathbf{k}$. Let $n \ge 1$; it is enough to obtain $\Sigma_n(K,\Gamma)$ as a finite union of sets $P \times Q$ with $P \subseteq K^n$, and $Q \subseteq \Gamma^n$ defined in Γ by a conjunction of atoms in the language of Γ-sets.

Suppose $(\vec{r},\vec{\gamma}) \in \Sigma_n(K,\Gamma)$. Then take a basis $b_1,\ldots,b_m$ of the $\mathbf{k}$-linear space $\mathbf{k}r_1 + \cdots + \mathbf{k}r_n$. Then $r_j = \sum_{i=1}^m a_{ij}b_i$ with $a_{ij} \in \mathbf{k}$, $j = 1,\ldots,n$, hence

$$(a_{i1},\ldots,a_{in},\vec{\gamma}) \in \Sigma_n(\mathbf{k},\Gamma) \quad \text{for } i = 1,\ldots,m.$$

Conversely, given any $a_{ij} \in \mathbf{k}$ $(1 \le i,j \le n)$ such that

$$(a_{i1},\ldots,a_{in},\vec{\gamma}) \in \Sigma_n(\mathbf{k},\Gamma) \quad \text{for } i = 1,\ldots,n$$

and any $b_1,\ldots,b_n \in K$ (not necessarily linearly independent over $\mathbf{k}$), we have $(\vec{r},\vec{\gamma}) \in \Sigma_n(K,\Gamma)$, where $r_j = \sum_{i=1}^n a_{ij}b_i$ for $j = 1,\ldots,n$. It follows that for all

$(\vec{r},\vec{\gamma}) \in K^n \times \Gamma^n$ we have: $(\vec{r},\vec{\gamma}) \in \Sigma_n(K,\Gamma)$ if and only if there is an $n \times n$-matrix $A = (a_{ij}) \in \mathbf{k}^{n^2}$ and a vector $\vec{b} \in K^n$ such that

$$(a_{i1},\dots,a_{in},\vec{\gamma}) \in \Sigma_n(\mathbf{k},\Gamma) \text{ for } i = 1,\dots,n, \text{ and } A\vec{b} = \vec{r}.$$

with $\vec{b}$ and $\vec{r}$ viewed as column vectors. By Lemma 4.2 the set of all $(A,\vec{\gamma}) \in \mathbf{k}^{n^2} \times \Gamma^n$, with $A = (a_{ij})$, such that the displayed condition holds, is a union $\bigcup_{\lambda\in\Lambda} P_\lambda \times Q_\lambda$ with finite Λ where each $P_\lambda \subseteq \mathbf{k}^{n^2}$, and each $Q_\lambda \subseteq \Gamma^n$ is defined in Γ by a conjunction of atoms in the language of Γ-sets.

For $\lambda \in \Lambda$, let P'_λ be the set of all $\vec{r} \in K^n$ for which there is a matrix $A \in P_\lambda$ and a vector $\vec{b} \in K^n$ such that $A\vec{b} = \vec{r}$. Then by the above, $\Sigma_n(K,\Gamma) = \bigcup_{\lambda\in\Lambda} P'_\lambda \times Q_\lambda$.

5.1. **Curious Mann pairs in positive characteristic.** Let p be a prime number, and let $\Omega \supseteq \mathbb{F}_p$ be of characteristic p so that every $\mathbb{F}_{p^e}$ with e a positive integer is a subfield of Ω. Let P be a finite nonempty set of prime numbers different from p, let $S(P)$ be the set of all positive integers all of whose prime factors are in P. Put

$$G := \{a \in \Omega^\times : \ a^N = 1 \text{ for some } N \in S(P)\},$$

an infinite subgroup of $\Omega^\times$ isomorphic to the direct product of the Prüfer groups $\mathbb{Z}(l^\infty)$ with $l \in P$.

Theorem 8.9 in [5] says that G has the Mann property, but this is also a consequence of earlier results in [2] and [15]. We now strengthen this as follows. The proof in [5] starts with a certain large enough finite field $\mathbb{F}_{p^e}$ where e is a positive integer. Put

$$f := e \cdot \prod_{l\notin P} l^\infty, \quad g := e \cdot \prod_{l\in P} l^\infty \qquad \text{(products of supernatural numbers).}$$

Then $\mathbb{F}_{p^e}(G) = \mathbb{F}_{p^g} \subseteq \Omega$, and so the infinite field $\mathbb{F}_{p^f}$ is linearly disjoint from $\mathbb{F}_{p^e}(G)$ over $\mathbb{F}_{p^e}$, so by (4) above we have a Mann pair $(\mathbb{F}_{p^f}, G)$ with the curious property that $\mathbb{F}_{p^f}(G) = \mathbb{F}_p^{\mathrm{ac}}$.

5.2. **Some Mann pairs in mixed characteristic.** Let p be a prime number and $E \supseteq \mathbb{F}_p$ a perfect field of characteristic p. Let $W[E] \supseteq W[\mathbb{F}_p] = \mathbb{Z}_p$ be the ring of Witt vectors over E, let $W(E) \supseteq W(\mathbb{F}_p) = \mathbb{Q}_p$ be its fraction field, and let $\tau : E^\times \to W(E)^\times$ be the Teichmüller lifting. It was shown in [5], 8.4, that $\tau(E^\times)$ satisfies a strong form of the Mann property. In fact:

Proposition 5.2. *$\big(\mathbb{Q}_p, \tau(E^\times)\big)$ is a Mann pair.*

Proof. Consider first the case $E = \mathbb{F}_p^{\mathrm{ac}}$. Then

$$\tau(E^\times) = \mathbb{U}[p'] := \{x \in W(E) : \ x^n = 1 \text{ for some } n \text{ not divisible by } p\}.$$

By arguments as in the proof of Theorem 1 in [12] one shows that $(\mathbb{Q}_p, \mathbb{U}[p'])$ is a Mann pair. (Alternatively, $(\mathbb{Q}, \mathbb{U}[p'])$ is a Mann pair by Theorem 1 in [12], and it remains to use (4) and the fact that $\mathbb{Q}_p$ and $\mathbb{Q}(\mathbb{U}[p'])$ are linearly disjoint over $\mathbb{Q}$.)

In the general case, we first arrange that E is algebraically closed so that $E \supseteq \mathbb{F}_p^{\mathrm{ac}}$ and $W(E) \supseteq W(\mathbb{F}_p^{\mathrm{ac}})$. Then $E^\times = (\mathbb{F}_p^{\mathrm{ac}})^\times G$ where the subgroup G of $E^\times$ is torsion-free. Then $\tau(E^\times) = \mathbb{U}[p']\tau(G)$. By the proof of 8.4 in [5], whenever $\gamma_1,\dots,\gamma_n \in \tau(G)$ are multiplicatively independent, they are algebraically independent over $W(\mathbb{F}_p^{\mathrm{ac}})$. Then Lemma 2.3 yields that $\big(W(\mathbb{F}_p^{\mathrm{ac}}), \tau(G)\big)$ is a Mann pair. Since $(\mathbb{Q}_p, \mathbb{U}[p'])$ is a Mann pair, we obtain from (3) that $(\mathbb{Q}_p, \tau(E^\times))$ is a Mann pair. □

6. Proof of Theorem 1.1

In this section we assume that Ω has characteristic zero, and, besides $\mathbf{k}$, we also let E, F, K be subfields of Ω. The following contains Proposition 5.16 of [5] as a special case, with almost the same proof.

Lemma 6.1. *Let E be a subfield of K such that all p^{th} roots of unity in K are in E, for every prime p. Let G be a pure subgroup of $E^\times$, and put*

$$H := \{h \in K^\times : \ h^d \in G \text{ for some positive integer } d\}.$$

Then for $a_1, \dots, a_n \in E^\times$, the equation $a_1x_1 + \cdots + a_nx_n = 1$ has the same non-degenerate solutions in G as in H.

Proof. By Lemma 5.15 of [5], it is enough to show the following.

Claim. Let p be a prime number, and let $h \in K^\times$ be such that $h^p = g \in G$ and $h \notin G$. Then $X^p - g \in E[X]$ is irreducible, and $Gh^{\mathbb{Z}}$, the subgroup of $K^\times$ generated by G and h, is pure in $E(h)^\times$.

Proof of the Claim. Since $h \notin G$ and K and E have the same p^{th} roots of unity, g is not a p^{th} power in $E^\times$. Thus by Theorem 9.1 from [9] the polynomial $X^p - g$ is irreducible in $E[X]$. To show that $Gh^{\mathbb{Z}} = \bigcup_{i=0}^{p-1} Gh^i$ is pure in $E(h)^\times$, suppose towards a contradiction that $f \in E(h)^\times$ and $f^d \in Gh^{\mathbb{Z}}$ where d is an integer greater than 1, but $f \notin Gh^{\mathbb{Z}}$. We can reduce to the case that d is prime and $f^m \notin G$ for $1 \le m < d$. So by Theorem 9.1 of [9] again, $X^d - f^d$ is irreducible in $E[X]$. As $E(f) \subseteq E(h)$ and $[E(h) : E] = p$, we get $d = p$. Let ζ be a primitive p^{th} root of unity in the algebraic closure of K. Then $E(h) \cap E(\zeta) = E$ and $E(h, \zeta)$ is a cyclic extension of degree p of $E(\zeta)$. Let $\sigma \in \operatorname{Gal}\big(E(h,\zeta)|E(\zeta)\big)$ be given by $\sigma(h) = \zeta h$. Then $\sigma(f) = \zeta^k f$ with $0 < k < p$. With $f = c_0 + c_1h + \cdots + c_{p-1}h^{p-1}$ and all $c_i \in E$ this gives

$$\begin{aligned}\sigma(f) &= c_0 + \zeta c_1 h + \cdots + \zeta^{p-1}c_{p-1}h^{p-1}\\ &= \zeta^k c_0 + \zeta^k c_1 h + \cdots + \zeta^k c_{p-1}h^{p-1}.\end{aligned}$$

This forces $c_i = 0$ for all $i \neq k$, so $f = c_k h^k$, as desired. □

Lemma 6.2. *Suppose $\mathbf{k}$ is algebraically closed, $F \supseteq \mathbf{k}$, $\operatorname{trdeg}(F|\mathbf{k}) = 1$, $\mathbf{k}^\times \cap \Gamma = \{1\}$, and $\Gamma \subseteq F^\times$ has finite rank. Then $(\mathbf{k}, \Gamma)$ is a Mann pair.*

Proof. We can assume that $\Gamma \neq \{1\}$. Consider the divisible closure

$$H := \{f \in F^\times : \ f^d \in \mathbf{k}^\times\Gamma \text{ for some positive integer } d\},$$

of $\mathbf{k}^\times\Gamma$ in $F^\times$. Now, as an abelian group, $\mathbf{k}^\times$ is divisible and hence injective, so we can take a subgroup $\Gamma' \supseteq \Gamma$ of H such that $\mathbf{k}^\times \cap \Gamma' = \{1\}$ and $\mathbf{k}^\times\Gamma' = H$. It follows that Γ' is pure in $F^\times$ and Γ'/Γ is a torsion group. Replacing Γ by Γ' we arrange that Γ is pure in $F^\times$. Take a finitely generated subgroup Γ_0 of Γ such that Γ/Γ_0 is a torsion group. Then $E := \mathbf{k}(\Gamma_0)$ is a function field over $\mathbf{k}$ of one variable. Take a finite subset S of $\mathcal{R}(E|\mathbf{k})$ such that all elements of Γ_0 are S-units, and put $\Gamma_1 := E^\times \cap \Gamma$. Since Γ_1/Γ_0 is a torsion group, all elements of Γ_1 are also S-units. Hence Γ_1 is isomorphic to a subgroup of $\bigoplus_{v \in S} \mathbb{Z}v$, and is thus finitely generated. Then by Corollary 3.2, $(\mathbf{k}, \Gamma_1)$ is a Mann pair. Since Γ is pure in $F^\times$ and $\mathbf{k}$ contains all roots of unity, Γ_1 is pure in $E^\times$. Now apply Lemmas 6.1 and 2.1 to get that $(\mathbf{k}, \Gamma)$ is a Mann pair. □

Theorem 6.3. *Suppose* $\mathbf{k}$ *is algebraically closed,* $\mathbf{k}^\times \cap \Gamma = \{1\}$*, and* Γ *has finite rank. Then* $(\mathbf{k}, \Gamma)$ *is a Mann pair.*

Proof. Let F be the algebraic closure of $\mathbf{k}(\Gamma)$ in Ω, and put

$$H := \{f \in F^\times : \ f^d \in \mathbf{k}^\times\Gamma \text{ for some positive integer } d\},$$

the divisible closure of $\mathbf{k}^\times\Gamma$ in $F^\times$. Now, as an abelian group, $\mathbf{k}^\times$ is divisible and hence injective, so we can take a subgroup $\Gamma' \supseteq \Gamma$ of H such that $\mathbf{k}^\times \cap \Gamma' = \{1\}$ and $\mathbf{k}^\times\Gamma' = H$. It follows that Γ' is divisible and Γ'/Γ is a torsion group. So replacing Γ by Γ' if necessary we can assume in the rest of the proof that Γ is divisible. Let $m := \operatorname{trdeg}(\mathbf{k}(\Gamma)|\mathbf{k})$, and take a chain

$$\mathbf{k} = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_m = F$$

of algebraically closed subfields of Ω such that for $i = 0, \dots, m-1$ we have $\operatorname{trdeg}(K_{i+1}|K_i) = 1$. Put

$$\Gamma_i := \Gamma \cap K_i^\times, \quad i = 1, \dots, m.$$

We claim that $(\mathbf{k}, \Gamma_i)$ is a Mann pair for $i = 1, \dots, m$. For $i = 1$ this holds by Lemma 6.2, since $\mathbf{k}(\Gamma_1) \subseteq K_1$ and K_1 has transcendence degree 1 over $\mathbf{k}$. Suppose $1 \le i < m$ and $(\mathbf{k}, \Gamma_i)$ is a Mann pair. Since Γ_i is divisible we have a subgroup Δ of Γ_{i+1} such that $\Gamma_{i+1} = \Gamma_i\Delta$ and $\Gamma_i \cap \Delta = \{1\}$. Then Δ has finite rank and $K_i^\times \cap \Delta = \{1\}$, and $K_i(\Delta) \subseteq K_{i+1}$, so (K_i, Δ) is a Mann pair. It follows from (3) in Section 5 that then $(\mathbf{k}, \Gamma_i\Delta) = (\mathbf{k}, \Gamma_{i+1})$ is a Mann pair.

This finishes the inductive proof of the claim. For $i = m$ this claim yields the desired conclusion. □

In this theorem we cannot replace the finite rank assumption by the weaker condition that Γ has the Mann property, as the following example shows. Take distinct indeterminates $a_1, x_1, a_2, x_2, a_3, x_3, \dots$ and put

$$\mathbf{k} := \mathbb{Q}(a_1, a_2, a_3, \dots)^{\mathrm{ac}}, \qquad K := \mathbf{k}(x_1, x_2, x_2, \dots).$$

Let Γ be the subgroup of $K^\times$ generated by $x_1, y_1, x_2, y_2, x_3, y_3, \dots$ where $y_i := 1 - a_i x_i$. Then $\mathbf{k}$ is algebraically closed, $\mathbf{k}^\times \cap \Gamma = \{1\}$, but $(\mathbf{k}, \Gamma)$ is not a Mann pair, since $a_i x_i + y_i = 1$ for all i. On the other hand, Γ has the Mann property because $x_1, y_1, x_2, y_2, \dots$ are algebraically independent over $\mathbb{Q}$ (but not over $\mathbf{k}$).

The condition $\mathbf{k}^\times \cap \Gamma = \{1\}$ in the theorem can be relaxed:

Corollary 6.4. *Suppose* $\mathbf{k}$ *is algebraically closed,* $\mathbf{k}^\times \cap \Gamma$ *is finite and* Γ *has finite rank. Then* $(\mathbf{k}, \Gamma)$ *is a Mann pair.*

Proof. Put $d := |\mathbf{k}^\times \cap \Gamma|$. The hypothesis implies that then the subgroup $\Delta := \{\gamma^d : \ \gamma \in \Gamma\}$ of Γ is of finite rank and has finite index in Γ, and that $\mathbf{k}^\times \cap \Delta = \{1\}$. Hence by the theorem, $(\mathbf{k}^{\mathrm{ac}}, \Delta)$ is a Mann pair, and thus $(\mathbf{k}, \Gamma)$ is a Mann pair by item (2) of the previous section. □

Another proof. The referee mentioned that Theorem 6.3 can be obtained from [7]. Here is a sketch of how. With the assumptions of Theorem 6.3 we can arrange in addition that Ω is equipped with a derivation making it a differentially closed field with $\mathbf{k}$ as its field of constants. Below, "definable", "Morley rank", and "Morley degree" are with respect to Ω^{eq}, with Ω viewed as a differentially closed field.

The derivation of Ω yields the logarithmic derivative map ld as a definable group morphism

$$x \mapsto x'/x \; : \; \mathbb{G}_{\mathrm{m}} = \Omega^{\times} \to \mathbb{G}_{\mathrm{a}} = \Omega,$$

with kernel $\mathbf{k}^{\times}$. Let $\gamma_1, \dots, \gamma_r \in \Gamma$ with $r \in \mathbb{N}$ generate a subgroup Γ_0 such that Γ/Γ_0 is a torsion group. Then

$$\mathrm{ld}(\Gamma) \subseteq \mathbb{Q} \cdot \mathrm{ld}(\gamma_1) + \cdots + \mathbb{Q} \cdot \mathrm{ld}(\gamma_r) \subseteq \mathbf{k} \cdot \mathrm{ld}(\gamma_1) + \cdots + \mathbf{k} \cdot \mathrm{ld}(\gamma_r) =: V.$$

Put $G := \mathrm{ld}^{-1}(V)$, a definable subgroup of $\mathbb{G}_{\mathrm{m}}$ of finite Morley rank with $\mathbf{k}^{\times} \subseteq G$ and $\Gamma \subseteq G$. Let $n \geq 2$ and let $X \subseteq G^n$ be the set of nondegenerate solutions of the equation $x_1 + \cdots + x_n = 0$ in G; by Lemma 2.1 it is enough to obtain a finite set $F(n) \subseteq G^n$ such that

$$X \subseteq \{(c_1 g_1, \dots, c_n g_n) : \; c_1, \dots, c_n \in \mathbf{k}^{\times}, \; (g_1, \dots, g_n) \in F(n)\}.$$

The semiabelian variety $\mathbb{G}_{\mathrm{m}}^n$ and its diagonal algebraic subgroup $\Delta := \{(g, \dots, g) : g \in \mathbb{G}_{\mathrm{m}}\}$ are defined over $\mathbf{k}$, which gives the semiabelian variety $A := \mathbb{G}_{\mathrm{m}}^n/\Delta$ over $\mathbf{k}$. Let $\bar{G} \subseteq A$ be the image of G^n under the natural map $\mathbb{G}_{\mathrm{m}}^n \to A$, so $\bar{G}$ is a definable subgroup of A of finite Morley rank containing $A(\mathbf{k})$. For $x \in G^n$, let $\bar{x}$ be its image in $\bar{G}$, and let $\bar{X} := \{\bar{x} : \; x \in X\} \subseteq \bar{G}$. It is now an exercise to show that each definable subset of $\bar{X}$ of Morley degree 1 has trivial model-theoretic stabilizer in $\bar{G}$, as defined in [7]. This allows us to apply Corollary 2.7 of [7] inductively to obtain that $\bar{X}$ is covered by finitely many cosets of $A(\mathbf{k})$; this gives the finite set $F(n) \subseteq G^n$ as desired.

This proof does not seem to have the same kind of effectiveness as the proof based on the Brownawell-Masser height bound.

7. Definable sets in Mann pairs

Let $\mathcal{L}$ be the language of rings augmented by two distinct unary relation symbols. Let T be the $\mathcal{L}$-theory whose models are the structures $(\Omega, \mathbf{k}, \Gamma)$ where Ω is an algebraically closed field with a subfield $\mathbf{k}$, and a subgroup Γ of $\Omega^{\times}$. Let $\mathcal{L}_{\Sigma}^{\mathrm{f,g}}$ be the 2-sorted language, with sorts f, g (the *field sort* and the *group sort*), and with the following nonlogical symbols:

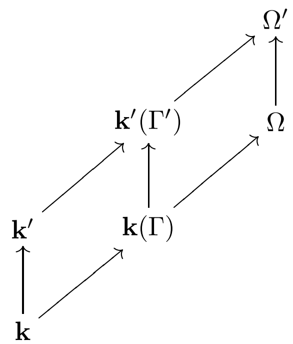
- constant symbols 0 and 1 of sort f,
- a unary function symbol $-$ of sort $(\mathrm{f};\mathrm{f})$,
- binary function symbols $+$ and $\cdot$ of sort $(\mathrm{f},\mathrm{f};\mathrm{f})$,
- a constant symbol 1 of sort g,
- a unary function symbol $^{-1}$ of sort $(\mathrm{g};\mathrm{g})$
- a binary function symbol $\cdot$ of sort $(\mathrm{g},\mathrm{g};\mathrm{g})$,
- for each $n \geq 1$, a $2n$-ary relation symbol Σ_n of sort $(\mathrm{f}, \dots, \mathrm{f}, \mathrm{g}, \dots, \mathrm{g})$ (n places of sort f and n places of sort g).

Of course, "1 used as a symbol of sort f" is different from "1 used as a symbol of sort g" and likewise with the multiplication symbol. For a model $(\Omega, \mathbf{k}, \Gamma)$ of T we construe $(\mathbf{k}, \Gamma)$ as an $\mathcal{L}_{\Sigma}^{\mathrm{f,g}}$-structure by interpreting the symbols in the obvious way; in particular, each Σ_n is interpreted as the previously defined $\Sigma_n(\mathbf{k}, \Gamma) \subseteq \mathbf{k}^n \times \Gamma^n$.

In the next result we do *not* assume that $(\mathbf{k}, \Gamma)$ is a Mann pair.

Lemma 7.1. *Let* $(\Omega, \mathbf{k}, \Gamma) \models T$. *Then every subset of* $\mathbf{k}^m \times \Gamma^n$ *definable in* $(\Omega, \mathbf{k}, \Gamma)$ *is definable in the* $\mathcal{L}_{\Sigma}^{\mathrm{f,g}}$*-structure* $(\mathbf{k}, \Gamma)$.

Proof. Take an $|\Omega|^+$-saturated elementary extension $(\Omega', \mathbf{k}', \Gamma')$ of $(\Omega, \mathbf{k}, \Gamma)$. It is easy to check that then $\mathbf{k}'(\Gamma')$ and Ω are linearly disjoint over $\mathbf{k}(\Gamma)$:



Let $\vec{a}, \vec{b} \in (\mathbf{k}')^m$ and $\vec{\alpha}, \vec{\beta} \in (\Gamma')^n$ be such that

$$\mathrm{tp}_{(\mathbf{k}',\Gamma')}\big((\vec{a},\vec{\alpha})|(\mathbf{k},\Gamma)\big) \;=\; \mathrm{tp}_{(\mathbf{k}',\Gamma')}\big((\vec{b},\vec{\beta})|(\mathbf{k},\Gamma)\big).$$

It suffices to prove that then

$$\mathrm{tp}_{(\Omega',\mathbf{k}',\Gamma')}\big((\vec{a},\vec{\alpha})|(\Omega,\mathbf{k},\Gamma)\big) \;=\; \mathrm{tp}_{(\Omega',\mathbf{k}',\Gamma')}\big((\vec{b},\vec{\beta})|(\Omega,\mathbf{k},\Gamma)\big).$$

The assumption on $(\vec{a},\vec{\alpha})$ and $(\vec{b},\vec{\beta})$ gives an automorphism of $(\mathbf{k}',\Gamma')$ over $(\mathbf{k},\Gamma)$ that takes $(\vec{a},\vec{\alpha})$ to $(\vec{b},\vec{\beta})$. This automorphism preserves the Σ-relations, so it extends to a field automorphism of $\mathbf{k}'(\Gamma')$ over $\mathbf{k}(\Gamma)$, which extends further to a field automorphism of Ω' over Ω by linear disjointness of $\mathbf{k}'(\Gamma')$ and Ω over $\mathbf{k}(\Gamma)$. This yields the desired equality of types. □

The implication (3) ⇒ (2) of Theorem 1.2 is trivial, so in view of Lemma 4.2 and Proposition 4.5, this theorem will be established once we prove the part (1) ⇒ (3), which is the next result.

Proposition 7.2. *Suppose $(\Omega,\mathbf{k},\Gamma) \models T$ and $(\mathbf{k},\Gamma)$ is a Mann pair. Then every subset of $\mathbf{k}^m \times \Gamma^n$ that is definable in $(\Omega,\mathbf{k},\Gamma)$ is a finite union of sets $X \times Y$ with $X \subseteq \mathbf{k}^m$ definable in the field $\mathbf{k}$ and $Y \subseteq \Gamma^n$ definable in the group Γ.*

Proof. As in the proof of the previous lemma we take an $|\Omega|^+$-saturated elementary extension $(\Omega',\mathbf{k}',\Gamma')$ of $(\Omega,\mathbf{k},\Gamma)$ and tuples $\vec{a},\vec{b} \in (\mathbf{k}')^m$ and $\vec{\alpha},\vec{\beta} \in (\Gamma')^n$ such that

$$\mathrm{tp}_{\mathbf{k}'}(\vec{a}|\mathbf{k}) \;=\; \mathrm{tp}_{\mathbf{k}'}(\vec{b}|\mathbf{k}) \text{ and } \mathrm{tp}_{\Gamma'}(\vec{\alpha}|\Gamma) \;=\; \mathrm{tp}_{\Gamma'}(\vec{\beta}|\Gamma).$$

By Lemma 7.1 it is enough to show that then

$$\mathrm{tp}_{(\mathbf{k}',\Gamma')}\big((\vec{a},\vec{\alpha})|(\mathbf{k},\Gamma)\big) \;=\; \mathrm{tp}_{(\mathbf{k}',\Gamma')}\big((\vec{b},\vec{\beta})|(\mathbf{k},\Gamma)\big).$$

The assumption on $\vec{a}$ and $\vec{b}$ gives an automorphism σ of $\mathbf{k}'$ over $\mathbf{k}$ such that $\sigma(\vec{a}) = \vec{b}$, and the assumption on $\vec{\alpha}$ and $\vec{\beta}$ gives an automorphism ϕ of Γ' over Γ such that $\phi(\vec{\alpha}) = \vec{\beta}$. It remains to show that this gives an automorphism (σ,ϕ) of the $\mathcal{L}_{\Sigma}^{\mathrm{f,g}}$-structure $(\mathbf{k}',\Gamma')$. This in turn reduces to establishing the following: Let N be a positive integer and $\vec{c}\,' \in (\mathbf{k}')^N$ and $\vec{\gamma}' \in (\Gamma')^N$. Then

$$(\mathbf{k}',\Gamma') \models \Sigma_N(\vec{c}\,',\vec{\gamma}') \iff (\mathbf{k}',\Gamma') \models \Sigma_N(\sigma(\vec{c}\,'),\phi(\vec{\gamma}')).$$

We prove the forward implication. (The backward implication follows in the same way.) Assume $(\mathbf{k}',\Gamma') \models \Sigma_N(\vec{c}\,',\vec{\gamma}')$. We have $\Sigma_N(\mathbf{k},\Gamma) = \bigcup_{i\in I} P_i \times Q_i$ where I is finite, each $P_i \subseteq \mathbf{k}^N$ is definable in the field $\mathbf{k}$ and each $Q_i \subseteq \Gamma^N$ is definable in the group Γ. Then $\Sigma_N(\mathbf{k}', \Gamma') = \bigcup_{i\in I} P_i' \times Q_i'$ where $P_i' \subseteq (\mathbf{k}')^N$ is defined in $\mathbf{k}'$ by any formula with parameters from $\mathbf{k}$ that defines P_i in the field $\mathbf{k}$, and $Q_i' \subseteq (\Gamma')^N$ is defined in Γ' by any formula with parameters from Γ that defines Q_i in the group Γ. Take $i \in I$ such that $\vec{c}\,' \in P_i'$ and $\vec{\gamma}' \in Q_i'$. It is clear that then $\sigma(\vec{c}\,') \in P_i'$ and $\phi(\vec{\gamma}') \in Q_i'$, so $(\mathbf{k}', \Gamma') \models \Sigma_N(\sigma(\vec{c}\,'), \phi(\vec{\gamma}'))$, as desired. □

8. The elementary theory of Ω with a Mann pair

In this section we discuss various model-theoretic properties of models $(\Omega, \mathbf{k}, \Gamma)$ of T where $(\mathbf{k}, \Gamma)$ is a Mann pair.

8.1. **Elementary equivalence and smallness.** We first prove that the theory of a model $(\Omega, \mathbf{k}, \Gamma)$ of T is determined by the $\mathcal{L}_\Sigma^{\mathrm{f,g}}$-theory of $(\mathbf{k}, \Gamma)$ whenever $\mathbf{k} \cup \Gamma$ is small in Ω. For this we do not need $(\mathbf{k}, \Gamma)$ to be a Mann pair. (A definition and basic properties of *small* are in [5], Section 2.)

Lemma 8.1. *Let $(\Omega_1, \mathbf{k}_1, \Gamma_1)$ and $(\Omega_2, \mathbf{k}_2, \Gamma_2)$ be models of T such that $\mathbf{k}_i \cup \Gamma_i$ is small in Ω_i for $i = 1, 2$, and $(\mathbf{k}_1, \Gamma_1) \equiv (\mathbf{k}_2, \Gamma_2)$ as $\mathcal{L}_\Sigma^{\mathrm{f,g}}$-structures. Then $(\Omega_1, \mathbf{k}_1, \Gamma_1) \equiv (\Omega_2, \mathbf{k}_2, \Gamma_2)$.*

Proof. It is harmless to assume CH (the continuum hypothesis), so we can reduce to the case that $(\Omega_i, \mathbf{k}_i, \Gamma_i)$ is saturated of cardinality $\aleph_1$ for $i = 1, 2$. Then we have an $\mathcal{L}_\Sigma^{\mathrm{f,g}}$-isomorphism

$$(\iota_{\mathrm{f}}, \iota_{\mathrm{g}}) : (\mathbf{k}_1, \Gamma_1) \to (\mathbf{k}_2, \Gamma_2).$$

This yields a ring isomorphism $\mathbf{k}_1[\Gamma_1] \to \mathbf{k}_2[\Gamma_2]$ extending both ι_{f} and ι_{g}, and hence a field isomorphism

$$\iota : \mathbf{k}_1(\Gamma_1)^{\mathrm{ac}} \to \mathbf{k}_2(\Gamma_2)^{\mathrm{ac}}.$$

By the smallness assumption the transcendence degree of Ω_i over $\mathbf{k}_i(\Gamma_i)^{\mathrm{ac}}$ is $\aleph_1$ for $i = 1, 2$. Thus we can extend ι to a field isomorphism $\Omega_1 \to \Omega_2$, which is then an $\mathcal{L}$-isomorphism. □

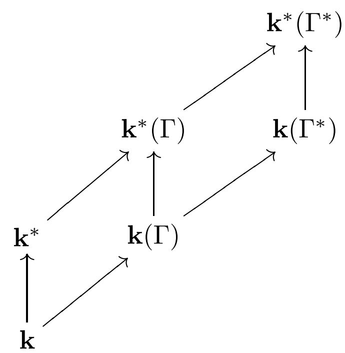
Let $\mathbf{k}$ be a subfield of the algebraically closed field Ω. A theorem of E. Artin says that if $1 < [\Omega : \mathbf{k}] < \infty$, then $\mathbf{k}$ is real closed with $\Omega = \mathbf{k}(\sqrt{-1})$. As noted at the end of Section 2 in [5] it follows that by work of Keisler [8] the following are equivalent:

(1) $\mathbf{k}$ is small in Ω;
(2) $[\Omega : \mathbf{k}] > 2$;
(3) $[\Omega : \mathbf{k}] = \infty$.

We use this equivalence in the proof of the next lemma, which is analogous to Lemma 6.1 in [5]. (The proofs are not at all similar.)

Lemma 8.2. *Assume $(\Omega, \mathbf{k}, \Gamma) \models T$ where $[\Omega : \mathbf{k}] > 2$ and $(\mathbf{k}, \Gamma)$ is a Mann pair. Then $\mathbf{k} \cup \Gamma$ is small in Ω.*

Proof. We can assume that Γ is infinite. Take a proper elementary extension $(\Omega^*, \mathbf{k}^*, \Gamma^*)$ of $(\Omega, \mathbf{k}, \Gamma)$. Using that $(\mathbf{k}, \Gamma)$ is a Mann pair, it follows easily that the subfields $\mathbf{k}^*$ and $\mathbf{k}(\Gamma^*)$ of Ω^* are linearly disjoint over $\mathbf{k}$. See the diagram below where all arrows are inclusions:



It follows that $\mathbf{k}^*(\Gamma)$ and $\mathbf{k}(\Gamma^*)$ are linearly disjoint over $\mathbf{k}(\Gamma)$. Among the subsets of Γ^* that are multiplicatively independent over Γ, take one, say B, that is maximal. It follows from Lemma 5.12 in [5] that B is algebraically independent over $\mathbf{k}(\Gamma)$. Hence B is algebraically independent over $\mathbf{k}^*(\Gamma)$. The maximality property of B guarantees that $B \neq \emptyset$ and for every $\gamma \in \Gamma^*$ there is a positive integer d with $\gamma^d \in \mathbf{k}^*(\Gamma)(B)$. Since $\mathbf{k}^*(\Gamma)(B)$ is a non-trivial purely transcendental extension of $\mathbf{k}^*(\Gamma)$ it follows that $\mathbf{k}^*(\Gamma^*, \sqrt{-1})$ is not algebraically closed, and so $\mathbf{k}^*(\Gamma^*, \sqrt{-1}) \neq \Omega^*$. Hence by the remark preceding this lemma $\mathbf{k}^* \cup \Gamma^*$ is small in Ω^*, and thus $\mathbf{k} \cup \Gamma$ is small in Ω; see Section 2 in [5]. □

8.2. **Stability.**

Proposition 8.3. *Let* $(\Omega, \mathbf{k}, \Gamma) \models T$ *be such that* $\mathbf{k}$ *is algebraically closed and* $(\mathbf{k}, \Gamma)$ *is a Mann pair. Then* $(\Omega, \mathbf{k}, \Gamma)$ *is stable, and if* Γ *is divisible, then* $(\Omega, \mathbf{k}, \Gamma)$ *is* ω*-stable.*

Proof. Take an infinite cardinal κ such that the abelian group Γ is κ-stable. We show that then $(\Omega, \mathbf{k}, \Gamma)$ is κ-stable. We can assume $\mathbf{k} \neq \Omega$, and that $|\Omega| = \kappa$. Take a κ^+-saturated elementary extension $(\Omega', \mathbf{k}', \Gamma')$ of $(\Omega, \mathbf{k}, \Gamma)$.

By the proofs of Lemma 7.1 and Proposition 7.2, the type of an element of $\mathbf{k}'$ over Ω in $(\Omega', \mathbf{k}', \Gamma')$ is determined by its type over $\mathbf{k}$ in the field $\mathbf{k}'$. Likewise, the type of an element of Γ' over Ω in $(\Omega', \mathbf{k}', \Gamma')$ is determined by its type over Γ in the group Γ'.

Now let $t \in \Omega(\mathbf{k}' \cup \Gamma')^{\mathrm{ac}}$, say $t \in \Omega(\vec{a}, \vec{\gamma})^{\mathrm{ac}}$ with $\vec{a} \in (\mathbf{k}')^m$ and $\vec{\gamma} \in (\Gamma')^n$. Then $\mathrm{tp}_{(\Omega', \mathbf{k}', \Gamma')}(t|\Omega)$ is determined by $\mathrm{tp}_{\mathbf{k}'}(\vec{a}|\mathbf{k})$, $\mathrm{tp}_{\Gamma'}(\vec{\gamma}|\Gamma)$ and the specification of a polynomial $P(X, Y, T) \in \Omega[X, Y, T]$ where $X = (X_1, \dots, X_m)$, $Y = (Y_1, \dots, Y_n)$ and T is a single indeterminate such that $P(\vec{a}, \vec{\gamma}, T) \in \Omega(\vec{a}, \vec{\gamma})[T]$ is irreducible and $P(\vec{a}, \vec{\gamma}, t) = 0$.

Finally, by the last argument of the proof of Lemma 8.1, all elements of Ω' outside $\Omega(\mathbf{k}' \cup \Gamma')^{\mathrm{ac}}$ realize the same type in $(\Omega', \mathbf{k}', \Gamma')$ over Ω.

Hence we have at most κ many different 1-types in $(\Omega', \mathbf{k}', \Gamma')$ over Ω. □

Remark. One can show that if $(\Omega, \mathbf{k}, \Gamma)$ is a model of T and $(\mathbf{k}, \Gamma)$ is a Mann pair, then the subsets $\mathbf{k}$ and Γ of Ω are definable in the structure $(\Omega, \mathbf{k} \cup \Gamma)$. Using this fact, Proposition 8.3 also follows from Fact 2.1 and Theorem 4.8 in [1].

8.3. **Axiomatizing** $(\Omega, \mathbf{k}, \Gamma)$**.** Our goal here is to show that if $(\Omega, \mathbf{k}, \Gamma)$ is a model of T and $(\mathbf{k}, \Gamma)$ is a Mann pair with $\mathbf{k}$ small in Ω, then the elementary theory of $(\Omega, \mathbf{k}, \Gamma)$ is completely determined by the elementary theories of the field $\mathbf{k}$ and of the group Γ. We achieve this after adding names for enough elements of $\mathbf{k}$ and Γ to witness that $(\mathbf{k}, \Gamma)$ is a Mann pair.

Fix a model $(\Omega_0, \mathbf{k}_0, \Gamma_0)$ of T such that $(\mathbf{k}_0, \Gamma_0)$ is a Mann pair. For each $n \geq 2$, fix a finite subset $\Gamma_0(n)$ of Γ_0^n such that

$$\Sigma_n^{\mathrm{nd}}(\mathbf{k}_0, \Gamma_0) = \bigcup_{\vec{\gamma} \in \Gamma_0(n)} \Sigma_n^{\mathrm{nd}}(\mathbf{k}_0, \Gamma_0; \vec{\gamma}) \times \Gamma_0 \vec{\gamma},$$

and for each $\vec{\gamma} \in \Gamma_0(n)$, fix a basis $B(\vec{\gamma}) \subseteq \mathbf{k}_0^n$ of the $\mathbf{k}_0$-linear subspace $\Sigma_n(\mathbf{k}_0, \Gamma_0; \vec{\gamma})$ of $\mathbf{k}_0^n$. Let $\mathcal{L}(\mathbf{k}_0, \Gamma_0)$ be the language $\mathcal{L}$ augmented by names for the elements of $\mathbf{k}_0 \cup \Gamma_0$.

In the following definition and subsequent remarks $(\Omega, \mathbf{k}, \Gamma)$ ranges over models of T that contain $(\Omega_0, \mathbf{k}_0, \Gamma_0)$ as a substructure. We construe such $(\Omega, \mathbf{k}, \Gamma)$ as an $\mathcal{L}(\mathbf{k}_0, \Gamma_0)$-structure in the obvious way, and say that $(\Omega, \mathbf{k}, \Gamma)$ *satisfies the Mann-axioms of* $(\Omega_0, \mathbf{k}_0, \Gamma_0)$ if for each $n \geq 2$:

(1) $\Sigma_n^{\mathrm{nd}}(\mathbf{k}, \Gamma) = \bigcup_{\vec{\gamma} \in \Gamma_0(n)} \Sigma_n^{\mathrm{nd}}(\mathbf{k}, \Gamma; \vec{\gamma}) \times \Gamma \vec{\gamma}$;

(2) for each $\vec{\gamma} \in \Gamma_0(n)$, the $\mathbf{k}$-linear subspace $\Sigma_n(\mathbf{k}, \Gamma; \vec{\gamma})$ of $\mathbf{k}^n$ is generated by $B(\vec{\gamma})$;

Remarks. The reason for this terminology is that we have a set $\mathrm{Mann}(\mathbf{k}_0, \Gamma_0)$ of sentences in the language $\mathcal{L}(\mathbf{k}_0, \Gamma_0)$ such that for all $(\Omega, \mathbf{k}, \Gamma)$ as above,

$$(\Omega, \mathbf{k}, \Gamma) \models \mathrm{Mann}(\mathbf{k}_0, \Gamma_0)$$
$$\Longleftrightarrow$$
$$(\Omega, \mathbf{k}, \Gamma) \text{ satisfies the Mann axioms of } (\Omega_0, \mathbf{k}_0, \Gamma_0).$$

In particular, if $(\Omega, \mathbf{k}, \Gamma)$ is an elementary extension of $(\Omega_0, \mathbf{k}_0, \Gamma_0)$, then $(\Omega, \mathbf{k}, \Gamma)$ satisfies the Mann axioms of $(\Omega_0, \mathbf{k}_0, \Gamma_0)$.

Let $\mathcal{L}^{\mathrm{f}} := \{0, 1, -, +, \cdot\}$ be the one-sorted language whose symbols are those of $\mathcal{L}_{\Sigma}^{\mathrm{f,g}}$ involving only sort f, and let $\mathcal{L}^{\mathrm{f}}(\mathbf{k}_0)$ be $\mathcal{L}^{\mathrm{f}}$ augmented by names for the elements of $\mathbf{k}_0$. Let $\mathcal{L}^{\mathrm{g}} := \{1, {}^{-1}, \cdot\}$ be the one-sorted language whose symbols are those of $\mathcal{L}_{\Sigma}^{\mathrm{f,g}}$ involving only sort g, and let $\mathcal{L}^{\mathrm{g}}(\Gamma_0)$ be $\mathcal{L}^{\mathrm{g}}$ augmented by names for the elements of Γ_0. Let $n \geq 1$, let $x = (x_1, \ldots, x_n)$ be a tuple of distinct f-variables, and let $y = (y_1, \ldots, y_n)$ be a tuple of distinct g-variables. A careful look at the proof of Lemma 4.2 for $\vec{r} = (1, \ldots, 1)$ yields the following: there are quantifier-free $\mathcal{L}^{\mathrm{f}}(\mathbf{k}_0)$-formulas $\phi_1(x), \ldots, \phi_m(x)$ and quantifier-free $\mathcal{L}^{\mathrm{g}}(\Gamma_0)$-formulas $\psi_1(y), \ldots, \psi_m(y)$ such that for all $(\Omega, \mathbf{k}, \Gamma) \models \mathrm{Mann}(\mathbf{k}_0, \Gamma_0)$,

$$\Sigma_n(\mathbf{k}^n, \Gamma^n) = \bigcup_{i=1}^{m} \phi_i(\mathbf{k}^n) \times \psi_i(\Gamma^n).$$

This uniformity is crucial in the proof of the next result.

Theorem 8.4. *Let* $(\Omega_1, \mathbf{k}_1, \Gamma_1)$ *and* $(\Omega_2, \mathbf{k}_2, \Gamma_2)$ *be models of* T *such that*

(1) $[\Omega_1 : \mathbf{k}_1] > 2$ *and* $[\Omega_2 : \mathbf{k}_2] > 2$*;*

(2) $(\Omega_0, \mathbf{k}_0, \Gamma_0) \subseteq (\Omega_1, \mathbf{k}_1, \Gamma_1)$ *and* $(\Omega_0, \mathbf{k}_0, \Gamma_0) \subseteq (\Omega_2, \mathbf{k}_2, \Gamma_2)$*;*

(3) $(\Omega_1, \mathbf{k}_1, \Gamma_1)$ *and* $(\Omega_2, \mathbf{k}_2, \Gamma_2)$ *satisfy the Mann-axioms of* $(\Omega_0, \mathbf{k}_0, \Gamma_0)$.

Then: $(\Omega_1, \mathbf{k}_1, \Gamma_1) \equiv_{\mathbf{k}_0 \cup \Gamma_0} (\Omega_2, \mathbf{k}_2, \Gamma_2) \iff \mathbf{k}_1 \equiv_{\mathbf{k}_0} \mathbf{k}_2$ *and* $\Gamma_1 \equiv_{\Gamma_0} \Gamma_2$.

Proof. The forward direction is clear. For the converse, assume $\mathbf{k}_1 \equiv_{\mathbf{k}_0} \mathbf{k}_2$ and $\Gamma_1 \equiv_{\Gamma_0} \Gamma_2$. Without loss we can also assume that $(\Omega_1, \mathbf{k}_1, \Gamma_1)$ and $(\Omega_2, \mathbf{k}_2, \Gamma_2)$ are κ-saturated, where $\kappa > |\mathbf{k}_0| + |\Gamma_0|$.

We construct a back-and-forth system between the $\mathcal{L}_\Sigma^{\mathrm{f,g}}$-structures $(\mathbf{k}_1, \Gamma_1)$ and $(\mathbf{k}_2, \Gamma_2)$. Let $\mathcal{S}_1$ be the set of substructures $(\mathbf{k}_1', \Gamma_1')$ of $(\mathbf{k}_1, \Gamma_1)$ that extend $(\mathbf{k}_0, \Gamma_0)$ and have cardinality $< \kappa$ (so $\mathbf{k}_1'$ is a subring, not necessarily a subfield, of $\mathbf{k}_1$, and Γ_1' is a subgroup of Γ_1). Define $\mathcal{S}_2$ likewise, with $(\mathbf{k}_2, \Gamma_2)$ in place of $(\mathbf{k}_1, \Gamma_1)$, and let $\mathcal{I}$ be the set of isomorphisms

$$\iota = (\iota_{\mathrm{f}}, \iota_{\mathrm{g}}) : (\mathbf{k}_1', \Gamma_1') \to (\mathbf{k}_2', \Gamma_2')$$

where $(\mathbf{k}_i', \Gamma_i') \in \mathcal{S}_i$ for $i = 1, 2$, such that ι is the identity on $(\mathbf{k}_0, \Gamma_0)$, $\iota_{\mathrm{f}} : \mathbf{k}_1' \to \mathbf{k}_2'$ is a partial elementary map from $\mathbf{k}_1$ to $\mathbf{k}_2$, and $\iota_{\mathrm{g}} : \Gamma_1' \to \Gamma_2'$ is a partial elementary map from Γ_1 to Γ_2. It is clear that the identity map of $(\mathbf{k}_0, \Gamma_0)$ belongs to $\mathcal{I}$.

Next we show that $\mathcal{I}$ is a back-and-forth system. By symmetry it is enough to show that we can go forth. Let

$$\iota = (\iota_{\mathrm{f}}, \iota_{\mathrm{g}}) : (\mathbf{k}_1', \Gamma_1') \to (\mathbf{k}_2', \Gamma_2')$$

be in $\mathcal{I}$. Replacing $\mathbf{k}_1'$ and $\mathbf{k}_2'$ by their fraction fields inside $\mathbf{k}_1$ and $\mathbf{k}_2$ and extending ι_{f} accordingly, without changing ι_{g}, we arrange that $\mathbf{k}_1'$ and $\mathbf{k}_2'$ are subfields of $\mathbf{k}_1$ and $\mathbf{k}_2$.

Let $a \in \mathbf{k}_1 \setminus \mathbf{k}_1'$ be given. Using saturation we can take $b \in \mathbf{k}_2 \setminus \mathbf{k}_2'$ and a field isomorphism $j : \mathbf{k}_1'(a) \to \mathbf{k}_2'(b)$ that extends ι_{f}, sends a to b, and is a partial elementary map from $\mathbf{k}_1$ to $\mathbf{k}_2$. It is obvious that $(\mathbf{k}_1'(a), \Gamma_1') \in \mathcal{S}_1$ and $(\mathbf{k}_2'(b), \Gamma_2') \in \mathcal{S}_2$, and it follows easily from the remark just before the lemma that $(j, \iota_{\mathrm{g}}) \in \mathcal{I}$.

Now let $\alpha \in \Gamma_1 \setminus \Gamma_1'$. By saturation we can find $\beta \in \Gamma_2 \setminus \Gamma_2'$ and a group isomorphism $h : \alpha^{\mathbb{Z}}\Gamma_1' \to \beta^{\mathbb{Z}}\Gamma_2'$ from Γ_1 to Γ_2 that extends ι_{g}, sends α to β, and is partial elementary map from Γ_1 to Γ_2. It is obvious that $(\mathbf{k}_1', \alpha^{\mathbb{Z}}\Gamma_1') \in \mathcal{S}_1$ and $(\mathbf{k}_2', \beta^{\mathbb{Z}}\Gamma_2') \in \mathcal{S}_2$. It follows easily from the remark just before the lemma that $(\iota_{\mathrm{f}}, h) : (\mathbf{k}_1', \alpha^{\mathbb{Z}}\Gamma_1') \to (\mathbf{k}_2', \beta^{\mathbb{Z}}\Gamma_2')$ belongs to $\mathcal{I}$.

We have now shown that $\mathcal{I}$ is a nonempty back-and-forth system. It follows that $(\mathbf{k}_1, \Gamma_1) \equiv_{(\mathbf{k}_0, \Gamma_0)} (\mathbf{k}_2, \Gamma_2)$. From (1), (3), and Lemma 8.2 we obtain that $\mathbf{k}_i \cup \Gamma_i$ is small in Ω_i for $i = 1, 2$. Then a proof like that of Lemma 8.1 yields

$$(\Omega_1, \mathbf{k}_1, \Gamma_1) \equiv_{\mathbf{k}_0 \cup \Gamma_0} (\Omega_2, \mathbf{k}_2, \Gamma_2) \iff \mathbf{k}_1 \equiv_{\mathbf{k}_0} \mathbf{k}_2 \text{ and } \Gamma_1 \equiv_{\Gamma_0} \Gamma_2,$$

as desired. □

For algebraically closed $\mathbf{k}_i$ this result takes the following form:

Corollary 8.5. *Let $(\Omega_1, \mathbf{k}_1, \Gamma_1)$ and $(\Omega_2, \mathbf{k}_2, \Gamma_2)$ be models of T such that*

(1) *$\mathbf{k}_1$ and $\mathbf{k}_2$ are algebraically closed, $\mathbf{k}_1 \neq \Omega_1$, $\mathbf{k}_2 \neq \Omega_2$;*
(2) *$(\Omega_0, \mathbf{k}_0, \Gamma_0) \subseteq (\Omega_1, \mathbf{k}_1, \Gamma_1)$ and $(\Omega_0, \mathbf{k}_0, \Gamma_0) \subseteq (\Omega_2, \mathbf{k}_2, \Gamma_2)$;*
(3) *$(\Omega_1, \mathbf{k}_1, \Gamma_1)$ and $(\Omega_2, \mathbf{k}_2, \Gamma_2)$ satisfy the Mann axioms of $(\Omega_0, \mathbf{k}_0, \Gamma_0)$.*

Then: $(\Omega_1, \mathbf{k}_1, \Gamma_1) \equiv_{\mathbf{k}_0 \cup \Gamma_0} (\Omega_2, \mathbf{k}_2, \Gamma_2) \iff \Gamma_1 \equiv_{\Gamma_0} \Gamma_2.$

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