EXPANSIONS OF THE GROUP OF INTEGERS BY BEATTY SEQUENCES

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Abstract. We study the model theoretic structure $(\mathbb{Z}, +, P_r)$ where $r > 1$ is an irrational number and the elements of P_r are of the form $\lfloor nr \rfloor$ for some $n \in \mathbb{Z} \setminus \{0\}$. We axiomatize of this structure and prove a quantifier elimination result. As a consequence, we get that definable subsets are not sparse unless they are finite. We also prove that there are no reducts of this structure expanding $(\mathbb{Z}, +)$.

1. Introduction

We investigate the expansion of the abelian group of integers by the following subset:

$$P_r := \Big\{ \lfloor nr \rfloor : n \in \mathbb{Z} \setminus \{0\} \Big\},$$

where $r > 1$ is an irrational number. The number 0 is taken out for some technical reasons and of course it has no effect on our results. There is a considerable difference between this expansion and the expansion by $P_r^+ := P_r \cap \mathbb{N}$, because $(\mathbb{Z}, +, P_r^+)$ defines the ordering of $\mathbb{Z}$, whereas we show in Corollary 5.10 that $(\mathbb{Z}, +, P_r)$ does not define the ordering. We also know that P_r is not definable in $(\mathbb{Z}, +, <)$, since it is shown in [4] that the only reduct of $(\mathbb{Z}, +, <)$ defining addition is $(\mathbb{Z}, +)$.

There have been some work on expansions of $(\mathbb{Z}, +)$ by a predicate. Earliest one is [11] by Moosa-Scanlon; even though the main focus of that paper is quite different than ours, the main result (Theorem A) there provides some stable expansions $(\mathbb{Z}, +)$. The papers [13, 12, 5, 9] by Poizat, Palacín-Sklinos, Conant and Lambotte-Point are more recent. In all these, the predicate is *sparse*, with certain different but similar meanings of the word sparse. A common sparsity property is that if we enumerate the predicate in an increasing way as $(a_n)_{n\in\mathbb{N}}$, then $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} > 1$. Conant calls a predicate satisfying this property *lacunary* in [3]. In the same paper, he proves that the expansion of $(\mathbb{Z}, +)$ by an infinite subset A of a finitely generated submonoid of $(\mathbb{N}_+, \cdot)$ is superstable of U-rank ω. It is possible that such an A is non-lacunary. For instance, if the finitely generated monoid contains

a, b with irrational $\log_b a$, then it is non-lacunary (see the paragraph after Theorem 3.16 in the Introduction of [3].)

Our study of the expansion $(\mathbb{Z}, +, P_r)$ is complimentary to the work [13, 12, 5, 9], as the set P_r is certainly not sparse: If $k \in P_r$, then the next element in P_r is either $k + \lfloor r \rfloor$ or $k + \lfloor r \rfloor + 1$. (This is also the reason of $(\mathbb{Z}, +, P_r^+)$ defining the ordering.) Actually, we prove that no infinite subset of $\mathbb{Z}$ definable in $(\mathbb{Z}, +, P_r)$ is sparse in the sense of the following definition.

Definition 1.1. Let $X \subseteq \mathbb{Z}$. We say that X has the *uniform gaps property* if there is a positive integer M such that

$$X \cap \{x + 1, \ldots, x + M\} \neq \emptyset$$

for every $x \in X$.

(A set having this property appears in different settings as that set being *syndetic*; see for instance [10]. Also the concept of *piecewise syndeticity* plays a central role in [9].)

Theorem 1.2. *Let $X \subseteq \mathbb{Z}$ be infinite and definable in $(\mathbb{Z}, +, P_r)$. Then X has the uniform gaps property.*

The proof of this result goes through a quantifier elimination result. In order to state that, let $L_\pm$ be the extension of the language $\{+, -, 0, 1\}$ of abelian groups with a distinguished element by unary predicate symbols $D_{m,+}$ and $D_{m,-}$ for each $m \geq 1$. We interpret the new symbols in $\mathbb{Z}$ as follows:

$$D^{\mathbb{Z}}_{m,+} := \{x \in \mathbb{Z} : x = my \text{ for some } y \in P_r\},$$

and

$$D^{\mathbb{Z}}_{m,-} := \{x \in \mathbb{Z} : x = my \text{ for some } y \notin P_r\}.$$

Note that $D^{\mathbb{Z}}_{1,+} = P_r$ and that $D^{\mathbb{Z}}_{m,+} \cup D^{\mathbb{Z}}_{m,-} = m\mathbb{Z}$.

Theorem 1.3. *The structure $\left(\mathbb{Z}, +, 0, 1, (D^{\mathbb{Z}}_{m,+})_{m\geq 1}, (D^{\mathbb{Z}}_{m,-})_{m\geq 1}\right)$ has quantifier elimination.*

It is more natural to consider our structure in the language $L_P = \{+, -, 0, 1, P\}$, where P is a unary predicate. Identifying P with $D_{1,+}$, we may think of L_P as a sublanguage of $L_\pm$.

Given $\vec{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ and $I \subseteq \{1, \ldots, n\}$, consider the following L_P-formula in variables $\vec{x} = (x_1, \ldots, x_n)$:

$$\exists y \left(\bigwedge_{i \in I} x_i + k_i y \in P \wedge \bigwedge_{j \notin I} x_j + k_j y \notin P \right).$$

By Theorem 1.3 above, there is a quantifier-free $L_{\pm}$-formula $\psi_{\vec{k},I}(\vec{x})$ such that

$$(\star)\forall x_1 \cdots \forall x_n \left(\exists y \left(\bigwedge_{i\in I} x_i + k_i y \in P \wedge \bigwedge_{j\notin I} x_j + k_j y \notin P \right) \leftrightarrow \psi_{\vec{k},I}(\vec{x}) \right).$$

holds in $\left(\mathbb{Z}, +, 0, 1, (D^{\mathbb{Z}}_{m,+})_{m\geq 1}, (D^{\mathbb{Z}}_{m,-})_{m\geq 1}\right)$. It turns out that $\psi_{\vec{k},I}$ can be chosen to be a quantifier-free L_P-formula; see the beginning of Section 4 for more details.

With this notation at hand, we have the following axiomatization.

Theorem 1.4. *Let $\mathcal{M} = \left(M, +, -, 0, 1, P^{\mathcal{M}}\right)$ be an L_P-structure. Then $\mathcal{M}$ is elementarily equivalent to $(\mathbb{Z}, +, -, 0, 1, P_r)$ if and only if the following hold:*

(1) $(M, +, -, 0, 1) \equiv (\mathbb{Z}, +, -, 0, 1)$,
(2) *for every $k_1, \ldots, k_n \in \mathbb{Z}$ and $I \subseteq \{1, \ldots, n\}$ the sentence $(\star)$ holds in $\mathcal{M}$,*
(3) *$k \in P_r$ if and only if $k \in P^{\mathcal{M}}$, for every $k \in \mathbb{Z}$.*

The technical parts of the proofs are done in an isomorphic structure: Let

$$\Gamma_r := \left\{ \exp(\frac{n2\pi i}{r}) \in \mathbb{C} : n \in \mathbb{Z} \right\}.$$

So $\Gamma_r = h(\mathbb{Z})$ where h is the group isomorphism sending n to $\exp(\frac{n2\pi i}{r})$. Being a subgroup of the unit circle, Γ_r has an *orientation* on it; the precise definition is given in the next section. Then the image of P_r under h becomes an *orientation interval*; see equality (3.2) below. This makes it easier to work in Γ_r and the notations get simpler. For this reason, in Section 2, we recall some facts about the circle and its subgroups.

In Section 3, we introduce Beatty Sequences and prove a few results about them to be used in the model theoretic arguments.

A back-and-forth system constructed in Section 4 is used to prove the theorems mentioned above in the rest of that section and Section 5.

The paper [14] by Tran-Walsberg has quite a bit of overlap with our work. The authors consider $\mathbb{Z}$ equipped with a 'cyclic ordering'; which in turn is the same as considering an infinite cyclic subgroup of the circle equipped with the orientation. We elaborate on this in Section 5.

In the last section, we prove a result analogous to the main result of [4]: there are no intermediate structures between $(\mathbb{Z}, +)$ and $(\mathbb{Z}, +, P_r)$.

Notations and Conventions. The set $\mathbb{N}$ of natural numbers contains 0 and $\mathbb{N}_+ = \mathbb{N} \setminus \{0\}$. We let the letters m, n, k, l vary in $\mathbb{Z}$, and if m is in $\mathbb{N}$ (or $\mathbb{N}_+$), we simply write $m \geq 0$ (or $m > 0$).

For $n > 0$, we denote the set $\{1, \ldots, n\}$ as $[n]$.

For a real number a, we use the notation $\lfloor a \rfloor$ for the largest integer smaller than or equal to a and $\{a\}$ denotes the difference $a - \lfloor a \rfloor$. (There will not be any occasions where this could be confused with the singleton containing a.)

2. The Circle and Its Subgroups

In the next section, we work with an infinite cyclic subgroup of the unit circle $\mathbb{S} := \{\beta \in \mathbb{C} : |\beta| = 1\}$. One may study such a group in the generality of *oriented abelian groups* (as in [7]) or *cyclically ordered abelian groups* (as in [6]). However there is no need to do so for our purposes.

Here we recall some generalities about $\mathbb{S}$ and at the end we say a few words about its subgroups. Let

$$e : \mathbb{R} \to \mathbb{S}, \quad e(x) := \exp(2\pi i x).$$

This is a surjective group homomorphism with kernel $\mathbb{Z}$.

We equip $\mathbb{S}$ with the *counter-clockwise orientation*: Given $\alpha, \beta, \gamma \in \mathbb{S}$, the relation $\mathcal{O}(\alpha, \beta, \gamma)$ holds if and only if there are $x, y, z \in \mathbb{R}$ such that $\alpha = e(x), \beta = e(y), \gamma = e(z)$, $x < y < z$, and $z - x < 1$.

If we fix $\alpha \in \mathbb{S}$, then we get a linear ordering $\mathcal{O}_\alpha(\cdot, \cdot) := \mathcal{O}(\alpha, \cdot, \cdot)$ on $\mathbb{S} \setminus \{\alpha\}$. For $\alpha = 1$, we denote the ordering of $\mathbb{S} \setminus \{1\}$ by the usual ordering sign:

$$\beta < \gamma \iff \mathcal{O}(1, \beta, \gamma).$$

Clearly, if $\alpha < \beta$ and $\beta < \gamma$, then $\mathcal{O}(\alpha, \beta, \gamma)$. So there is no harm in writing

$$\alpha_1 < \alpha_2 < \cdots < \alpha_{n-1} < \alpha_n$$

when $\alpha_1 < \alpha_2, \alpha_2 < \alpha_3, \ldots, \alpha_{n-1} < \alpha_n$.

We extend the definition of $<$ to all of $\mathbb{S}$ by setting $1 < \alpha$ for $\alpha \neq 1$.

The relation of orientation and the group operation is as follows:

$$\mathcal{O}(\alpha, \beta, \gamma) \iff \mathcal{O}(\alpha\delta, \beta\delta, \gamma\delta)$$

and

$$\mathcal{O}(\alpha, \beta, \gamma) \iff \mathcal{O}(\gamma^{-1}, \beta,^{-1} \alpha^{-1})$$

for every $\alpha, \beta, \gamma, \delta \in \mathbb{S}$.

The circle $\mathbb{S}$ has the topology induced by the Euclidean topology on $\mathbb{C}$ and a basis for this topology consists of *orientation intervals*: Given $\alpha, \beta \in \mathbb{S}$ we define the orientation interval determined by α and β to be

$$(\alpha, \beta) := \big\{\gamma \in \mathbb{S} : \mathcal{O}(\alpha, \gamma, \beta)\big\}.$$

We do not assume $\alpha < \beta$ for this definition. So both (α, β) and (β, α) are orientation intervals and they are disjoint. Such an interval is empty only when $\alpha = \beta$.

We define the *length* of an orientation interval $I = (\alpha, \beta)$ to be

$$l(I) := \beta\alpha^{-1}.$$

So if $\alpha = e(a)$ and $\beta = e(b)$, then the length of (α, β) is $e(b - a)$. In particular, the length of an orientation interval is 1 if and only if it is empty.

Below, we use the word *interval* to mean *orientation interval.* We also use the notations $[\alpha, \beta)$, $(\alpha, \beta]$, and $[\alpha, \beta]$ with the obvious meanings, and we call them *intervals* as well.

Proposition 2.1. *Let $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_m$ be distinct elements of $\mathbb{S}$. Then the following conditions are equivalent:*

(1) $\displaystyle\bigcap_{i=1}^{m}(\alpha_i, \beta_i) \neq \emptyset$.
(2) *there is $i_0 \in [m]$ such that $\alpha_{i_0} \in (\alpha_i, \beta_i)$ for every $i \neq i_0$.*
(3) *there is $j_0 \in [m]$ such that $\beta_{j_0} \in (\alpha_i, \beta_i)$ for every $i \neq j_0$.*

Moreover, if one of these conditions hold, then

$$\bigcap_{i=1}^{m}(\alpha_i, \beta_i) = (\alpha_{i_0}, \beta_{j_0}).$$

Proof. Suppose $\gamma \in (\alpha_i, \beta_i)$ for every $i \in [m]$. Then $\mathcal{O}_\gamma(\beta_i, \alpha_i)$ for each $i \in [m]$. Choose i_0 such that α_{i_0} is maximal among α_i with respect to $\mathcal{O}_\gamma$. Clearly, $\alpha_{i_0} \in (\alpha_i, \beta_i)$ for each $i \neq i_0$. Similarly, taking β_{j_0} to be minimum among β_i with respect to $\mathcal{O}_\gamma$ we see that $\beta_{j_0} \in (\alpha_i, \beta_i)$ for each $i \neq j_0$. So the first condition implies the others.

Conversely, assume that $\alpha_{i_0} \in (\alpha_i, \beta_i)$ for each $i \neq i_0$. Then

$$(\alpha_{i_0}, \beta_{j_0}) \subseteq \bigcap_{i=1}^{m}(\alpha_i, \beta_i),$$

where β_{j_0} is minimum among β_i with respect to $\mathcal{O}_{\alpha_{i_0}}$.

If $\beta_{j_0} \in (\alpha_i, \beta_i)$ for each $i \neq i_0$. Then we take α_{i_0} to be maximum among α_i with respect to $\mathcal{O}_{\beta j_0}$ in order to get

$$(\alpha_{i_0}, \beta_{j_0}) \subseteq \bigcap_{i=1}^{m}(\alpha_i, \beta_i).$$

The last sentence of the proposition follows from the rest of the proof above. $\square$

For $k \in \mathbb{N}_+$, let ζ_k denote the primitive k^{th} root of unity $e(\frac{1}{k})$.

Proposition 2.2. *Let $\alpha, \beta, \gamma \in \mathbb{S}$ and $k \in \mathbb{N}_+$ such that $l\big((\alpha, \gamma)\big) < \zeta_k$. Then $\beta^k \in (\alpha^k, \gamma^k)$ if and only if $\beta \in (\zeta_k^s\alpha, \zeta_k^s\gamma)$ for some $s \in \mathbb{Z}$.*

Proof. Note that if $\alpha = \gamma$, then the result is clear; so we assume $\alpha \neq \gamma$. We may also assume that $\alpha < \gamma$: if $\gamma < \alpha$, then we may consider $\alpha' = \zeta_k \alpha$ and $\gamma' = \zeta_k \gamma$ in the place of α and γ. Then $\alpha' < \gamma'$ by our assumption on the length of (α, γ), and the conclusion holds for α, γ if and only if it holds for α', γ'.

So let $0 \leq a < c < 1$ be such that $\alpha = e(a)$ and $\gamma = e(c)$. Note that $c - a < \frac{1}{k}$ by assumption. Also let $\beta = e(b)$ for some $0 \leq b < 1$.

Suppose $\beta^k \in (\alpha^k, \gamma^k)$ and take b' such that $\beta^k = e(b')$ and $ka < b' < kc$. Then $s := kb - b' \in \mathbb{Z}$ and we get

$$ka + s < kb < kc + s.$$

Dividing by k, we have

$$a + \frac{s}{k} < b < c + \frac{s}{k}.$$

Applying e, we get that $\beta \in (\zeta_k^s \alpha, \zeta_k^s \gamma)$.

Conversely, let $\beta \in (\zeta_k^s \alpha, \zeta_k^s \gamma)$ for some $s \in \mathbb{Z}$. In other words, $\beta \zeta_k^{-s} \in (\alpha, \gamma)$. Take b'' such that $\beta \zeta_k^{-s} = e(b'')$ and $a < b'' < c$. Then $t := b - b'' - \frac{s}{k} \in \mathbb{Z}$ and

$$a + t < b - \frac{s}{k} < c + t.$$

Now multiplying by k and applying e we get $\beta^k \in (\alpha^k, \gamma^k)$. $\square$

Note that the s in this proposition can be chosen among $0, 1, \ldots, k-1$.

Given $\alpha = e(x)$ with $0 \leq x < 1$ and $k \in \mathbb{N}_+$, we let $\alpha^{1/k}$ denote $e(\frac{x}{k})$; so $\alpha^{1/k}$ is the k^{th} root of α with the smallest argument. Note that $1^{1/k} = 1$ and that for $x \in \mathbb{R}$ we have $e(x)^{1/k} = e(\frac{x}{k})\zeta_k^{-\lfloor x \rfloor}$. In particular, if $\alpha = e(a)$, then $(\alpha^k)^{1/k} = \alpha \zeta_k^{-\lfloor ak \rfloor}$. However, we always have $(\alpha^{1/k})^k = \alpha$. The following observation will be useful.

Lemma 2.3. *Let $\alpha, \beta \in \mathbb{S}$, $k, l \in \mathbb{Z}$. Suppose that $\alpha^k = \beta$. Then $\alpha = \beta^{1/k} \zeta_k^l$ if and only if $\alpha \in (\zeta_k^l, \zeta_k^{l+1})$.*

Proof. Clear. $\square$

Using this new notation, the previous proposition has the following consequences.

Corollary 2.4. *Let $\alpha, \beta, \gamma \in \mathbb{S}$ and $k \in \mathbb{N}_+$. Suppose that $\gamma \not< \alpha$. Then $\beta^k \in (\alpha, \gamma)$ if and only if there is $s \in \{0, 1, \ldots, k-1\}$ such that $\beta \in \left(\zeta_k^s \alpha^{1/k}, \zeta_k^s \gamma^{1/k}\right)$.*

Corollary 2.5. *Let $\alpha, \beta, \gamma \in \mathbb{S}$ and $k \in \mathbb{N}_+$. Suppose that $\gamma < \alpha$. Then $\beta^k \in (\alpha, \gamma)$ if and only if there is $s \in \{0, 1, \ldots, k-1\}$ such that $\beta \in \left(\zeta_k^s \alpha^{1/k}, \zeta_k^{s+1} \gamma^{1/k}\right)$.*

Regularly Dense Oriented Abelian Groups. Let $\Gamma \leq \mathbb{S}$ be nontrivial. Then Γ is either finite or dense in $\mathbb{S}$. When it is finite, it consists of m^{th} roots of unity for some $m > 0$. When Γ is dense, it is indeed *regularly dense* in the following sense.

Proposition 2.6. *Let $\Gamma \leq \mathbb{S}$ be infinite. Then for any distinct $\alpha, \beta \in \Gamma$ and prime p, there is $\gamma \in \Gamma$ such that $\gamma^p \in (\alpha, \beta)$.*

For the proof of this, we refer the reader to Definition 8.1.7 in [7] and the remark succeeding it. Note that the conclusion of the proposition above is slightly different than the definition of regular density for oriented abelian groups in [7], but it is easy to see that they are indeed equivalent. It follows that for any $n > 0$ and distinct $\alpha, \beta \in \Gamma$, there is $\gamma \in \Gamma$ such that $\gamma^n \in (\alpha, \beta)$.

3. Beatty Sequences

Let $r > 1$ be an irrational number. The *Beatty Sequence generated by r* is $\mathcal{B}_r = (\lfloor mr \rfloor)_{m>0}$; we put $b_m = \lfloor mr \rfloor$. This is an increasing sequence and we let P_r^+ denote the set whose elements are the terms of $\mathcal{B}_r$.

A related sequence is $\mathcal{S}_r = \left(\lfloor \frac{n+1}{r} \rfloor - \lfloor \frac{n}{r} \rfloor\right)_{n>0}$; we put $s_n = \lfloor \frac{n+1}{r} \rfloor - \lfloor \frac{n}{r} \rfloor$. (For the relation between $\mathcal{B}_r$ and $\mathcal{S}_r$, see Lemma 3.1 below.) Note that $s_n \in \{0, 1\}$ for each $n > 0$. Actually, it is better to think of $\mathcal{S}_r$ as an infinite word in the alphabet $\{0, 1\}$. As such, it is called *the Characteristic Sturmian Word of Slope $\frac{1}{r}$*. It has the property that for every m, it has exactly $m + 1$ many different subwords of length m.

Both Beatty Sequences and Sturmian Words have rich theories that we do not get into here, and we refer the interested reader to [1]. We only need the following connection between $\mathcal{B}_r$ and $\mathcal{S}_r$ which is Lemma 9.1.3 of [1], but we include a proof for completeness.

Lemma 3.1. *Let $n \in \mathbb{N}_+$. Then $n \in P_r^+$ if and only if $s_n = 1$.*

Proof. Let $n \in P_r^+$. Then $n = \lfloor kr \rfloor$ for some $k \in \mathbb{N}_+$. So $kr - 1 < n < kr$ and after diving by r we have

$$k - \frac{1}{r} < \frac{n}{r} < k.$$

Therefore $\lfloor \frac{n}{r} \rfloor = k - 1$ and $\lfloor \frac{n+1}{r} \rfloor = k$. Thus $s_n = \lfloor \frac{n+1}{r} \rfloor - \lfloor \frac{n}{r} \rfloor = 1$. As all the implications are reversible we get the desired result. □

We would like to consider the negative elements as well; so we define

$$P_r = \left\{ \lfloor nr \rfloor : n \in \mathbb{Z} \setminus \{0\} \right\}.$$

For $m > 0$, we have $-m \in P_r$ if and only if $m - 1 \in P_r^+$. So

$$P_r = P_r^+ \cup (-P_r^+ - 1).$$

We also extend the definitions of b_n and s_n to all integers n.

Lemma 3.1 is actually correct for all $n \in \mathbb{Z}$:

$$n \in P_r \iff s_n = 1. \tag{3.1}$$

It is easy to see that $s_n = 1$ if and only if $\{\frac{n}{r}\} > 1 - \frac{1}{r}$. Putting this together with (3.1), for every $n \in \mathbb{Z}$, we get

$$n \in P_r \iff \{\frac{n}{r}\} > 1 - \frac{1}{r}. \tag{3.2}$$

Since r is irrational, the image of $\mathbb{Z}\frac{1}{r}$ under e is not finite, hence it is a dense subgroup of $\mathbb{S}$. Let Γ_r be that subgroup, and let h denote the map $n \mapsto e(\frac{n}{r})$. So we have an isomorphism of abelian groups with a distinguished element:

$$h : \big(\mathbb{Z}, +, -, 0, 1\big) \simeq \big(\Gamma_r, \cdot, ^{-1}, 1, h(1)\big).$$

By (3.2), the image of P_r under h is $\big(h(-1), 1\big) \cap \Gamma_r$. Therefore expanding $\mathbb{Z}$ by P_r is the same as expanding Γ_r by $\big(h(-1), 1\big) \cap \Gamma_r$.

Using Proposition 2.2 and its corollaries, we give a criterion for certain linear combinations of integers being in P_r in terms of intervals in $\mathbb{S}$.

Proposition 3.2. *Let $k \in \mathbb{N}_+$ and $a, c \in \mathbb{Z}$. Then $a + kc \in P_r$ if and only if there is $s \in \mathbb{Z}$ such that*

$$h(c) \in \big(h(-a-1)^{1/k}\zeta_k^s, h(-a)^{1/k}\zeta_k^{s+s_a}\big).$$

Proof. First, note that $a \in P_r$ if and only if $h(-a) < h(-a-1)$.

By (3.2) we have

$$a+kc \in P_r \iff h(a)h(c)^k \in \big(h(-1), 1\big) \iff h(c)^k \in \big(h(-a-1), h(-a)\big).$$

Now combining Corollaries 2.4 and 2.5 and using the first sentence of this proof, we get that

$$h(c)^k \in \big(h(-a-1), h(-a)\big)$$

if and only if there is $s \in \mathbb{Z}$ with

$$h(c) \in \big(h(-a-1)^{1/k}\zeta_k^s, h(-a)^{1/k}\zeta_k^{s+s_a}\big).$$

This gives the desired equivalence. □

Corollary 3.3. *Let $k \in \mathbb{N}_+$ and $a, c \in \mathbb{Z}$. Then $a + kc \notin P_r$ if and only if there is $s \in \mathbb{Z}$ such that*

$$h(c) \in \big[h(-a)^{1/k}\zeta_k^{s+s_a}, h(-a-1)^{1/k}\zeta_k^{s+1}\big].$$

Proof. Clear from the previous proposition. □

Next result will be useful in handling the cases when k is negative.

Lemma 3.4. *Let $a, c \in \mathbb{Z}$ and $k < 0$. Then $a + kc \in P_r$ if and only if $-a - 1 - kc \in P_r$.*

Proof. Clear from the fact that $\lfloor -x \rfloor = -\lfloor x \rfloor - 1$ for $x \notin \mathbb{Z}$. $\square$

Definition 3.5. For $a \in \mathbb{Z}$, $k \in \mathbb{N}_+$, and $s \in \{0, 1, \ldots, k-1\}$, let

$$\begin{aligned} U_{a,k,s} &:= \big(h(-a-1)^{1/k}\zeta_k^s, h(-a)^{1/k}\zeta_k^{s+s_a}\big), \\ V_{a,k,s} &:= \big[h(-a)^{1/k}\zeta_k^{s+s_a}, h(-a-1)^{1/k}\zeta_k^{s+1}\big]. \end{aligned}$$

Also let

$$U_{a,k} := \bigcup_{s=0}^{k-1} U_{a,k,s} \quad \text{and} \quad V_{a,k} := \bigcup_{s=0}^{k-1} V_{a,k,s}.$$

We extend the definitions to $k = 0$ as follows:

$$U_{a,0} := \begin{cases} \mathbb{S} & : \text{if } a \in P_r \\ \emptyset & : \text{if } a \notin P_r \end{cases} \quad \text{and} \quad V_{a,0} := \begin{cases} \emptyset & : \text{if } a \in P_r \\ \mathbb{S} & : \text{if } a \notin P_r \end{cases}$$

Finally, we let $\tilde{V}_{a,k}$ denote the interior of $V_{a,k}$.

With this notation in hand, Proposition 3.2 and Corollary 3.3 translate as follows: Given $k \in \mathbb{N}$ and $a, c \in \mathbb{Z}$ we have

$$a + kc \in P_r \iff h(c) \in U_{a,k}, \tag{3.3}$$

and

$$a + kc \notin P_r \iff h(c) \in V_{a,k}. \tag{3.4}$$

Lemma 3.6. *Let $a, b \in \mathbb{Z}$ and $k, l \in \mathbb{N}_+$. Suppose $g = \gcd(k, l)$ and write $k = gk'$ and $l = gl'$. Then the following hold.*

(1) *Suppose $\zeta_{k'} < h(1)$. Then there is $s \in \mathbb{Z}$ such that $h(a)^{1/k}\zeta_k^s \in U_{b,l}$.*
(2) *Suppose $h(1) < \zeta_{k'}$. Then there is $s \in \mathbb{Z}$ such that $h(a)^{1/k}\zeta_k^s \in U_{b,l}$ if and only if*

$$h(l'a + k'b) \in \big(h(-k'), 1\big).$$

Proof. Using Corollaries 2.4 and 2.5, $h(a)^{1/k}\zeta_k^s \in U_{b,l}$ if and only if

$$h(l'a)^{1/k'}\zeta_{k'}^{\lfloor \frac{la}{r} \rfloor - l\lfloor \frac{a}{r} \rfloor + sl'} \in \big(h(-b-1), h(-b)\big).$$

Since $\gcd(k', l') = 1$, there is $s \in \mathbb{Z}$ with $h(a)^{1/k}\zeta_k^s \in U_{b,l}$ if and only if there is $t \in \mathbb{Z}$ with $h(l'a)^{1/k'}\zeta_{k'}^t \in \big(h(-b-1), h(-b)\big)$.

If $\zeta_{k'} < h(1)$, then there is such a t, regardless of what a, b, l are.

If $h(1) < \zeta_{k'}$, then we may use Proposition 2.2 to conclude that there is $s \in \mathbb{Z}$ with $h(a)^{1/k}\zeta_k^s \in U_{b,l}$ if and only if

$$h(l'a) \in \big(h(-k'b - k'), h(-k'b)\big).$$

After simplification, this means $h(l'a + k'b) \in \big(h(-k'), 1\big)$.

$\square$

This proof can be modified to prove the next analogous result.

Lemma 3.7. *Let $a, b \in \mathbb{Z}$ and $k, l \in \mathbb{N}_+$. Suppose $g = \gcd(k, l)$ and write $k = gk'$ and $l = gl'$. Then the following hold.*

(1) *Suppose $\zeta_{k'} < h(-1)$. Then there is $s \in \mathbb{Z}$ such that $h(a)^{1/k}\zeta_k^s \in V_{b,l}$.*
(2) *Suppose $h(-1) < \zeta_{k'}$. Then there is $s \in \mathbb{Z}$ such that $h(a)^{1/k}\zeta_k^s \in V_{b,l}$ if and only if*

$$h(l'a + k'b') \in \big[1, h(-k')\big].$$

Let $\vec{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, $\vec{k} = (k_1, \ldots, k_n) \in \mathbb{N}^n$, $I \subseteq [n]$, and $J \subseteq [n] \setminus I$ be given. We define

$$V_{\vec{a},\vec{k},I,J} = \bigcap_{i \in I} U_{a_i,k_i} \cap \bigcap_{j \in J} (V_{a_j,k_j} \setminus \tilde{V}_{a_j,k_j}) \cap \bigcap_{j \in [n] \setminus (I \cup J)} \tilde{V}_{a_j,k_j}.$$

Note that $V_{a_j,k_j} \setminus \tilde{V}_{a_j,k_j}$ has $2k_j$ many points. Therefore $V_{\vec{a},\vec{k},I,J}$ is finite for $J \neq \emptyset$ and $V_{\vec{a},\vec{k},I,\emptyset}$ is an open subset of $\mathbb{S}$.

We record the following without proof.

Lemma 3.8. *Let $\vec{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, $\vec{k} = (k_1, \ldots, k_n) \in \mathbb{N}_+^n$, and $\emptyset \neq J \subseteq [n]$. Then $h(c) \in V_{\vec{a},\vec{k},I,J} \cap \Gamma_r$ if and only if there is a subset J' of J such that $c = \frac{-a_j}{k_j}$ for every $j \in J'$, $c = \frac{-a_j - 1}{k_j}$ for every $j \in J \setminus J'$, and $h(c) \in \bigcap_{i \in I} U_{a_i,k_i} \cap \bigcap_{j \in [n] \setminus (I \cup J)} \tilde{V}_{a_j,k_j}$.*

Definition 3.9. Let $\vec{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ and $I \subseteq [n]$. We say that $(a_1, \ldots, a_n) \in \mathbb{Z}^n$ *realizes the $(\vec{k}, I)$-pattern* if there is $c \in \mathbb{Z}$ such that

$$a_i + k_i c \in P_r \iff i \in I.$$

For $\vec{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n$, let $|\vec{k}| = (|k_1|, \ldots, |k_n|)$. Then using Lemma 3.4, $(a_1, \ldots, a_n) \in \mathbb{Z}^n$ realizes the $(\vec{k}, I)$-pattern if $(a'_1, \ldots, a'_n)$ realizes the $(|\vec{k}|, I)$-pattern where $a'_i = a_i$ for $k_i \geq 0$ and $a'_i = -a_i - 1$ for $k_i < 0$. Therefore, we may focus on the case that $\vec{k} \in \mathbb{N}^n$.

By (3.3) and (3.4), if $\vec{k} \in \mathbb{N}^n$, then $(a_1, \ldots, a_n) \in \mathbb{Z}^n$ realizes the $(\vec{k}, I)$-pattern if and only if

$$\bigcap_{i \in I} U_{a_i,k_i} \cap \bigcap_{j \notin I} V_{a_j,k_j} \cap \Gamma_r \neq \emptyset.$$

We may decompose the intersection above as

$$\left(\bigcap_{i \in I} U_{a_i,k_i} \cap \bigcap_{j \notin I} \tilde{V}_{a_j,k_j} \cap \Gamma_r \right) \cup \left(\bigcup_{\emptyset \neq J \subseteq [n] \setminus I} V_{\vec{a},\vec{k},I,J} \cap \Gamma_r \right).$$

Using Lemma 3.8, the finite component is under control.

Let's focus on $\bigcap_{i\in I} U_{a_i,k_i} \cap \bigcap_{j\notin I} \tilde{V}_{a_j,k_j} \cap \Gamma_r$. Since Γ_r is dense in $\mathbb{S}$, this set is nonempty if and only if the open set

$$V_{\vec{a},\vec{k},I,\emptyset} = \bigcap_{i\in I} U_{a_i,k_i} \cap \bigcap_{j\notin I} \tilde{V}_{a_j,k_j}$$

is nonempty.

Using Proposition 2.1, it is easy to see that this intersection is nonempty if and only if one of the following holds:

(1) there are $i_0 \in I$ and $s \in \mathbb{Z}$ such that

$$h(-a_{i_0} - 1)^{1/k_{i_0}} \zeta_{k_{i_0}}^s \in \bigcap_{i\in I, i\neq i_0} U_{a_i,k_i} \cap \bigcap_{j\notin I} \tilde{V}_{a_j,k_j},$$

(2) there are $j_0 \in [n] \setminus I$ and $s \in \mathbb{Z}$ such that

$$h(-a_{j_0})^{1/k_{j_0}} \zeta_{k_{j_0}}^s \in \bigcap_{i\in I} U_{a_i,k_i} \cap \bigcap_{j\notin I, j\neq j_0} \tilde{V}_{a_j,k_j}.$$

In order to summarize these observations, we make the following definitions: let $k, l \in \mathbb{N}$ with $g = \gcd(k,l)$ and $k' := k/g$, $l' := l/g$

$$A_{k,l} := \begin{cases} \mathbb{Z}\times\mathbb{Z} & : \text{if } \zeta_{k'} < h(1) \\ \big\{(a,b)\in\mathbb{Z}\times\mathbb{Z} : h(l'a+k'b) \in \big(h(-k'),1\big)\big\} & : \text{if } h(1) < \zeta_{k'} \end{cases}$$

$$B_{k,l} := \begin{cases} \mathbb{Z}\times\mathbb{Z} & : \text{if } \zeta_{k'} < h(-1) \\ \big\{(a,b)\in\mathbb{Z}\times\mathbb{Z} : h(l'a+k'b) \in \big(1,h(-k')\big)\big\} & : \text{if } h(-1) < \zeta_{k'} \end{cases}$$

Combining Lemmas 3.6 and 3.7 with the observations above, we obtain the following.

Proposition 3.10. *Let $\vec{k} = (k_1,\ldots,k_n) \in \mathbb{N}^n$ and $I \subseteq [n]$. Then $\vec{a} = (a_1,\ldots,a_n) \in \mathbb{Z}^n$ realizes the $(\vec{k}, I)$-pattern if and only if one of the following conditions holds:*

(1) $\bigcup_{\emptyset\neq J\subseteq[n]\setminus I} V_{\vec{a},\vec{k},I,J} \cap \Gamma_r \neq \emptyset$.
(2) *there is $i_0 \in I$ with $(-a_{i_0}-1, a_i) \in A_{k_{i_0},k_i}$ for every $i \in I\setminus\{i_0\}$ and $(-a_{i_0}-1, a_j) \in B_{k_{i_0},k_j}$ for every $j \in [n]\setminus I$.*
(3) *there is $j_0 \notin I$ with $(-a_{j_0}, a_i) \in A_{k_{j_0},k_i}$ for every $i \in I$ and $(-a_{j_0}, a_j) \in B_{k_{j_0},k_j}$ for every $j \in [n]\setminus(I\cup\{j_0\})$.*

The final result of this section expresses the interval $\big(h(k),1\big)$ in terms of $\big(h(-1),1\big)$ when $h(1) < \zeta_k$.

Lemma 3.11. *Let $k > 0$ be such that $h(1) < \zeta_k$. Then $\alpha \in \big(1, h(k)\big)$ if and only if*

$$\alpha^{-1} \in \bigcup_{i=0}^{k-1} \big(h(-i-1), h(-i)\big) \cup \{h(-1), h(-2), \ldots, h(-(k-1))\}.$$

(Note that the interval $\big(h(-i-1), h(-i)\big)$ is the same as $h(i)\big(h(-1), 1\big)$ with the natural definition of multiplying an interval by an element.)

Proof. The assumption $h(1) < \zeta_k$ gives $h(i-1) < h(i)$ for every $i \in [k]$. So we have the decomposition

$$\begin{aligned}\big(1, h(k)\big) &= \big(1, h(1)\big] \cup \big(h(1), h(2)\big] \cup \cdots \cup \big(h(k-1), h(k)\big)\\ &= \big(1, h(1)\big) \cup \cdots \cup \big(h(k-1), h(k)\big) \cup \{h(1), \ldots, h(k-1)\}\\ &= \bigcup_{i=0}^{k-1} h(i)\big(1, h(1)\big) \cup \{h(1), \ldots, h(k-1)\}\end{aligned}$$

This finishes the proof, since $\beta \in \big(1, h(1)\big)$ if and only if $\beta^{-1} \in \big(h(-1), 1\big)$. □

4. Expanding the Group of Integers

We would like to consider the model theoretic structure obtained by expanding the abelian group of integers by the subset P_r. We have seen above that $\mathbb{Z}$ is isomorphic as an abelian group with a subgroup Γ_r of $\mathbb{S}$ that happens to be dense in $\mathbb{S}$. The work in the previous section was mostly done in Γ_r, but it is straightforward to pull those results back to $\mathbb{Z}$ via the map h.

Let $L := \{+, -, 0, c\}$ be the language of abelian groups with a distinguished element c. Let T be the theory of the L-structure $(\mathbb{Z}, +, -, 0, 1)$.

We extend L to $L_P := L \cup \{P\}$ where P is a unary relation symbol. Our main objective is to study the L_P-structure

$$\mathfrak{Z} := \Big(\mathbb{Z}, +, -, 0, 1, P_r\Big).$$

For $\vec{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ and $I \subseteq [n]$, we define $\phi_{\vec{k},I}(x_1, \ldots, x_n)$ to be the following L_P-formula:

$$\exists y \left(\bigwedge_{i \in I} x_i + k_i y \in P \wedge \bigwedge_{j \notin I} x_j + k_j y \notin P \right).$$

Therefore for $(a_1, \ldots, a_n) \in \mathbb{Z}^n$, we have $\mathfrak{Z} \models \phi_{\vec{k},I}(a_1, \ldots, a_n)$ if and only if $(a_1, \ldots, a_n)$ realizes the $(\vec{k}, I)$-pattern. Our immediate aim is to show that $\phi_{\vec{k},I}(x_1, \ldots, x_n)$ is equivalent in $\mathfrak{Z}$ to a quantifier-free L_P-formula using the results of the previous section. First of all, by Lemma 3.4, it is enough to do this for $\vec{k} \in \mathbb{N}^n$. Below $\vec{x} = (x_1, \dots, x_n)$ is an n-tuple of distinct variables.

It is clear from Lemma 3.11 that for each $k, l \in \mathbb{N}$, the sets $A_{k,l}$ and $B_{k,l}$ are definable by quantifier-free L_P-formulas $\alpha_{k,l}(x, y)$ and $\beta_{k,l}(x, y)$. Using these formulas and Lemma 3.8, for a nonempty subset J of $[n] \setminus I$ there is a quantifier-free L_P-formula $\tau_{\vec{k},I,J}(\vec{x})$ such that for $\vec{a} \in \mathbb{Z}^n$ we have

$$\mathfrak{Z} \models \tau_{\vec{k},I,J}(\vec{a}) \iff V_{\vec{a},\vec{k},I,J} \cap \Gamma_r \neq \emptyset.$$

Let $\tau_{\vec{k},I}(\vec{x})$ be $\bigvee_{\emptyset \neq J \subseteq [n] \setminus I} \tau_{\vec{k},I,J}(\vec{x})$.

Let $\theta_{\vec{k},I}(\vec{x})$ be

$$\bigvee_{i_0 \in I} \left(\bigwedge_{i \in I \setminus \{i_0\}} \alpha_{k_{i_0},k_i}(-x_{i_0} - 1, x_i) \wedge \bigwedge_{j \in [n] \setminus I} \beta_{k_{i_0},k_j}(-x_{i_0} - 1, x_j) \right),$$

and similarly let $\chi_{\vec{k},I}(\vec{x})$ be

$$\bigvee_{j_0 \notin I} \left(\bigwedge_{i \in I} \alpha_{k_{j_0},k_i}(-x_{j_0}, x_i) \wedge \bigwedge_{j \in [n] \setminus (I \cup \{j_0\})} \beta_{k_{j_0},k_j}(-x_{j_0}, x_j) \right).$$

Now taking the disjunction of all these formulas, we obtain a quantifier-free L_P-formula $\psi_{\vec{k},I}(\vec{x})$:

$$\tau_{\vec{k},I}(\vec{x}) \vee \theta_{\vec{k},I}(\vec{x}) \vee \chi_{\vec{k},I}(\vec{x}),$$

and by Proposition 3.10 we have

$$\mathfrak{Z} \models \forall x_1 \cdots \forall x_n \big(\phi_{\vec{k},I}(x_1, \dots, x_n) \leftrightarrow \psi_{\vec{k},I}(x_1, \dots, x_n) \big)$$

Let T_r be the L_P-theory extending T by the condition above; namely for every $\vec{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n$ and $I \subseteq [n]$, we add the following sentence as an axiom:

$$\forall x_1 \cdots \forall x_n \left(\phi_{\vec{k},I}(x_1, \dots, x_n) \leftrightarrow \psi_{\vec{k},I}(x_1, \dots, x_n) \right)$$

We shall construct a back-and-forth system between certain substructures of models of T_r.

Let $\mathcal{M}$ and $\mathcal{N}$ be $\aleph_0$-saturated models of T_r. Let $S_{\mathcal{M}}$ be the collection of countable L_P-substructures $\mathcal{M}'$ of $\mathcal{M}$ such that M' is a pure subgroup of M. We define $S_{\mathcal{N}}$ in a similar way.

Note that the group $\mathbb{Z}$ has a copy in each member of $S_{\mathcal{M}}$ and $S_{\mathcal{N}}$ as the subgroup generated by the constant c. However, those copies may not be isomorphic as L_P-substructures of $\mathcal{M}$ and $\mathcal{N}$.

Let $\mathfrak{B}(\mathcal{M}, \mathcal{N})$ be the collection of L_P-isomorphisms $f : \mathcal{M}' \to \mathcal{N}'$, where $\mathcal{M}' \in S_{\mathcal{M}}$ and $\mathcal{N}' \in S_{\mathcal{N}}$.

Proposition 4.1. *For $\aleph_0$-saturated models $\mathcal{M}$ and $\mathcal{N}$ of T_r, the collection $\mathfrak{B}(\mathcal{M}, \mathcal{N})$ is a back-and-forth system.*

Proof. Let $f : \mathcal{M}' \to \mathcal{N}'$ be in $\mathfrak{B}(\mathcal{M}, \mathcal{N})$ and $\alpha \in M \setminus M'$. By symmetry, it suffices to extend f to an element of $\mathfrak{B}(\mathcal{M}, \mathcal{N})$ that contains α in its domain.

Let M'' be the pure subgroup of M generated by M' and α; namely:

$$M'' = \langle M' \cup \{\alpha\}\rangle_M := \{\gamma \in M : m\gamma \in M' \oplus \mathbb{Z}\alpha \text{ for some } m > 0\}.$$

Also let $\mathcal{M}''$ be the L_P-substructure of $\mathcal{M}$ with the underlying set M''. We would like to extend f to $\mathcal{M}''$. That amounts to finding $\beta \in N$ with the following property:

($*$) For every $a \in M'$, $k \in \mathbb{Z}$, $n > 0$, and $\gamma \in M$ if $a + k\alpha = n\gamma$, then there is $\delta \in N$ such that $f(a) + k\beta = n\delta$, and

$$\gamma \in P \iff \delta \in P.$$

This condition without the last part just means that $\langle M' \cup \{\alpha\}\rangle_M$ and $\langle N' \cup \{\beta\}\rangle_N$ are isomorphic as groups. Since the reducts of $\mathcal{M}$ and $\mathcal{N}$ to L are models of T, there is certainly such an element β in N. So the point is to find β in a way that that isomorphism of groups is indeed an L_P-isomorphism.

By saturation, it suffices to find $\beta \in N$ satisfying a given finite fragment of ($*$). So let $a_1, \dots, a_m \in M'$, $k_1, \dots, k_m \in \mathbb{Z}$, $n_1, \dots, n_m \in \mathbb{N}_+$, and $\gamma_1, \dots, \gamma_m \in M$ be such that $a_i + k_i\alpha = n_i\gamma_i$ for each i. Then we need to find $\beta, \delta_1, \dots, \delta_m \in N$ such that $f(a_i) + k_i\beta = n_i\delta_i$ and $\delta_i \in P$ if and only if $\gamma_i \in P$ for every i.

Let $\nu = \operatorname{lcm}(n_1, \dots, n_m)$, and let $d \in \{0, \dots, \nu - 1\}$ and $\alpha' \in M$ be such that $\alpha = d + \nu\alpha'$. Then $\gamma_i = a_i' + k_i'\alpha'$, where $n_i a_i' = a_i + k_i d$ and $k_i' = \frac{k\nu}{n_i}$. Since M' is pure in M, it contains a_i'. Therefore it suffices to find $\beta' \in N$ such that for every i:

$$a_i' + k_i'\alpha' \in P \iff f(a_i') + k_i'\beta' \in P$$

Taking $\vec{k} = (k_1', \dots, k_n')$ and $I = \{i : a_i' + k_i'\alpha' \in P\}$, we have

$$\mathcal{M} \models \phi_{\vec{k},I}(a_1', \dots, a_n').$$

Hence

$$\mathcal{M} \models \psi_{\vec{k},I}(a_1', \dots, a_n') \text{ and } \mathcal{M}' \models \psi_{\vec{k},I}(a_1', \dots, a_n').$$

Thus

$$\mathcal{N}' \models \psi_{\vec{k},I}\left(f(a_1'), \dots, f(a_n')\right) \text{ and } \mathcal{N} \models \psi_{\vec{k},I}\left(f(a_1'), \dots, f(a_n')\right).$$

As a result $\mathcal{N} \models \phi_{\vec{k},I}(f(a_1'), \dots, f(a_n'))$ and hence there is $\beta' \in N$ with the desired property:

$$f(a_i') + k_i'\beta' \in P \iff i \in I \iff a_i' + k_i'\alpha' \in P.$$

□

4.1. **Quantifier Elimination and Axiomatization.** The theory T_r does not have quantifier elimination for the obvious reason that for any $n > 1$, the definable subgroup consisting of elements divisible by n is not quantifier-free definable. However, we still do not get quantifier elimination after adding predicate symbols to represent those subgroups, because we also need to know whether the element obtained by dividing by n is in P or not. So for every $n \geq 1$ we add two new unary predicate symbols $D_{n,+}$ and $D_{n,-}$ to the language L_P to obtain $L_\pm$ and let $T_\pm$ be the definitional extension of T_r to an $L_\pm$-theory by adding the following for each $n \geq 1$:

$$\forall x\big(D_{n,+}(x) \leftrightarrow \exists y(x = ny \wedge y \in P)\big)$$
$$\forall x\big(D_{n,-}(x) \leftrightarrow \exists y(x = ny \wedge y \notin P)\big)$$

Therefore, every model $\mathcal{M}$ of T_r expands to a model of $T_\pm$; we still denote this extension by $\mathcal{M}$. Note that for a model $\mathcal{M}$ of $T_\pm$, we have $D_{n,+}(\mathcal{M}) \cup D_{n,-}(\mathcal{M}) = nM$ for every $n \geq 1$ and $D_{1,+}(\mathcal{M}) = P(\mathcal{M})$.

Now we are ready to prove Theorem 1.3 in a stronger form.

Theorem 4.2. *The theory $T_\pm$ has quantifier elimination.*

Proof. It suffices to prove the following:

(†) Let $\mathcal{M}$ and $\mathcal{N}$ be models of $T_\pm$ and $\mathcal{A}$ a common finitely generated $L_\pm$-substructure of $\mathcal{M}$ and $\mathcal{N}$. Then $\mathcal{M} \equiv_A \mathcal{N}$. (This means that $\mathcal{M}$ and $\mathcal{N}$ are elementarily equivalent as $L_\pm(A)$-structures)

(For why this is enough, see, for instance, Proposition 18.2 of [8].)

We may assume that $\mathcal{M}$ and $\mathcal{N}$ are $\aleph_0$-saturated. Let

$$M' = \langle A \rangle_M \text{ and } N' = \langle A \rangle_N.$$

Clearly, M' and N' are isomorphic as abelian groups via a map extending the identity map on A. If $n\alpha = a$ for some $a \in A$ and $\alpha \in M$, then there is $\beta \in N$ with $n\beta = a$. Then the isomorphism sends α to β. But we also have $\alpha \in P(\mathcal{M})$ if and only if $\beta \in P(\mathcal{N})$, since either both $\mathcal{M}$ and $\mathcal{N}$ satisfy $D_{n,+}(a)$ or they both satisfy $D_{n,-}(a)$. Therefore M' and N' are underlying sets of $L_\pm$-substructures $\mathcal{M}'$ and $\mathcal{N}'$ of $\mathcal{M}$ and $\mathcal{N}$ respectively, and they are isomorphic. Since M' and N' are countable and pure in M and N, that isomorphism is in $\mathfrak{B}(\mathcal{M}, \mathcal{N})$. It follows that $\mathcal{M} \equiv_{M'} \mathcal{N}$ and in particular $\mathcal{M} \equiv_A \mathcal{N}$. □

Given $\aleph_0$-saturated models $\mathcal{M}$ and $\mathcal{N}$ of T_r, we may still have that $\mathfrak{B}(\mathcal{M}, \mathcal{N}) = \emptyset$. So in order to get completeness we extend $T_\pm$ to T_r^* by adding $\mathbb{Z}$-*axioms*: Given $k \in \mathbb{N}_+$ if $k \in P_r$, then we add the axiom $k \in P$, otherwise we add the axiom $k \notin P$. (Recall that 1 is in the language, so $k = 1 + \cdots + 1$.) Clearly, T_r^* still has quantifier elimination.

With this extension, we get Theorem 1.4.

Theorem 4.3. *The theory T_r^* is complete.*

Proof. Clearly, $\mathfrak{Z}$ is an algebraically prime model of T_r^* . Since T_r^* has quantifier elimination, we get that T_r^* is complete. $\square$

Question. According to this theorem each T_r^* is a completion of T_r. Is it correct that each completion of T_r is given as the theory of an expansion of the group of integers by a Beatty Sequence?

5. Definable Sets

When dealing with the definable sets, we constantly switch between $\mathfrak{Z}$ and the isomorphic structure

$$\mathfrak{G} := \big(\Gamma_r, \cdot, ^{-1}, 1, h(1), P_r\big),$$

where P_r denotes the interval $\big(h(-1), 1\big) \cap \Gamma_r$ of Γ_r.

For a subset X of Γ_r^n and $m > 0$ we let

$$X^{(m)} := \big\{(x_1^m, \dots, x_n^m) : \vec{x} \in X\big\}, \text{ and}$$
$$X^{1/m} := \{\vec{y} \in \Gamma_r^n : (y_1^m, \dots, y_n^m) \in X\}.$$

For a subset A of $\mathbb{S}$, we define $[A]$ to be the subgroup of $\mathbb{S}$ generated by A and

$$\langle A \rangle := \{\alpha \in \mathbb{S} : \alpha^m \in [A] \text{ for some } m > 0\}.$$

We collect some easy facts about these notions.

Lemma 5.1. *Let $X, Y \subseteq \Gamma_r^n$ and $m > 0$.*

(1) $D_{m,+}^{1/m} = P_r$ *and* $P_r^{(m)} = D_{m,+}$.
(2) $(X^{(m)})^{1/m} = X$ *and* $(X^{1/m})^{(m)} = X \cap (\Gamma_r^n)^{(m)}$
(3) $(X \cap Y)^{1/m} = X^{1/m} \cap Y^{1/m}$, $(X \cup Y)^{1/m} = X^{1/m} \cup Y^{1/m}$, *and* $(\Gamma_r^n \setminus X)^{1/m} = \Gamma_r^n \setminus X^{1/m}$.
(4) *For* $\vec{a} \in (\{0, \dots, m-1\})^n$, *let* $X_{\vec{a}} = h(\vec{a})X \cap (\Gamma_r^n)^{(m)}$. *Then*
$$X = \bigcup_{\vec{a}} h(-\vec{a})X_{\vec{a}}.$$

Before analyzing definable sets in detail, we would like to clarify the connection with the paper [14].

Lemma 5.2. *Let $m > 0$ and let $X = (\alpha, \beta) \cap \Gamma_r^{(m)}$. Then any set $(\alpha\gamma_1, \beta\gamma_2) \cap \Gamma_r^{(m)}$ with $\gamma_1, \gamma_2 \in \Gamma_r^{(m)}$ is definable in $(\Gamma_r, \cdot, X)$.*

Proof. It suffices to show that $Y = (\alpha, \beta\gamma) \cap \Gamma_r^{(m)}$ is definable in $(\Gamma_r, \cdot, X)$, where $\gamma = \gamma_2\gamma_1^{-1}$.

Let $\lambda = \beta\alpha^{-1}$ and take $\delta \in (\lambda^{1/2}, \lambda) \cap \Gamma_r^{(m)}$. Suppose that n is the largest natural number such that $\gamma\delta^{-n} < \lambda$.

Then Y is the following union:

$$Y = X \cup \delta X \cup \dots \cup \delta^n X \cup ((\alpha\delta^{n+1}, \beta\gamma) \cap \Gamma_r^{(m)}),$$

Note that

$$(\alpha\delta^{n+1}, \beta\gamma) \cap \Gamma_r^{(m)} = \delta^{n+1}(X \cap \gamma(\delta^{n+1})^{-1}X).$$

So Y is definable in $(\Gamma_r, \cdot, X)$. $\square$

Letting $X = P_r$ in this lemma, we have the following consequence.

Corollary 5.3. *Any interval of Γ_r is definable in $\mathfrak{G}$.*

Therefore, the structures $\mathfrak{G}$ and $\big(\Gamma_r, \cdot, \mathcal{O}\big)$ are interdefinable, where $\mathcal{O}$ is the restriction of the orientation of $\mathbb{S}$ to Γ_r. In [14], the authors study the latter structure; or rather the pull-back of it under h. More precisely, they consider the structure $(\mathbb{Z}, +, C)$ where C is a *cyclic order* on $\mathbb{Z}$ (see the beginning of Section 2 of [14] for the precise definition). They prove that any such structure is dp-minimal; see Theorem 2.3 of [14]. Then they prove in Proposition 2.5 that $(\mathbb{Z}, +, C)$ is isomorphic either to $\big(\Gamma_r, \cdot, \mathcal{O}\big)$ for some r, or to one of two other possible expansions of $(\mathbb{Z}, +)$, which we prefer not to introduce here. As a result we get that $\big(\Gamma_r, \cdot, \mathcal{O}\big)$ has NIP, and hence $\mathfrak{G}$ has NIP. (It also follows from Corollary 4.3 of [2] that $\big(\Gamma_r, \cdot, \mathcal{O}\big)$ has NIP.)

We appeal to the topology on Γ_r^n induced from $\mathbb{S}^n$ in order to study definable sets. An open basis for that topology on Γ_r consists of sets of the form $I \cap \Gamma_r$ where I is an open interval of $\mathbb{S}$; below we refer to these sets as *convex* sets. From now on we use the word *interval* to mean *open interval of* Γ_r; so an interval is a convex set whose end points are in Γ_r.

The quantifier elimination result in the previous section gives a very simple characterization of definable sets in a model $\mathcal{M}$ of T_r^*: They are Boolean combinations of sets defined by formulas of the form

$$a + \vec{k}\vec{x} = 0, D_{m,+}(a + \vec{k}\vec{x}), \text{and } D_{m,-}(a + \vec{k}\vec{x})$$

where $\vec{x} = (x_1, \ldots, x_n)$ is a tuple of variables, $a \in M$ and $\vec{k} \in \mathbb{Z}^n$.

It is simpler in the sense that the negations of formulas $D_{m,+}(a + \vec{k}\vec{x})$ and $D_{m,-}(a + \vec{k}\vec{x})$ define sets that are finite unions of sets defined by the same kind of formulas. Also we would like to consider the formula

$$D_{m,-}(a + \vec{k}\vec{x}) \wedge a + \vec{k}\vec{x} \neq 0 \wedge a + \vec{k}\vec{x} \neq -m$$

in the place of $D_{m,-}(a + \vec{k}\vec{x})$.

So any definable set in $\mathfrak{G}$ is of the form

$$(*) \qquad \bigcup_{i=1}^{s} \bigcap_{j=1}^{t_i} X_{ij},$$

where each X_{ij} is one of the following forms

$$\{\vec{x} \in \Gamma_r^n : \vec{x}^{\vec{k}} h(a) = 1\} \tag{A}$$

$$\{\vec{x} \in \Gamma_r^n : \vec{x}^{\vec{k}} h(a) \neq 1\} \tag{B}$$

$$\{\vec{x} \in \Gamma_r^n : \vec{x}^{\vec{k}} h(a) \in P_r^{(m)}\} \tag{C}$$

$$\{\vec{x} \in \Gamma_r^n : \vec{x}^{\vec{k}} h(a) \in Q_r^{(m)}\} \tag{D}$$

where $\vec{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n \setminus \{\vec{0}\}$, $a \in \mathbb{Z}$, $m > 0$, $\vec{x}^{\vec{k}} := x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$, and $Q_r = (1, h(-1)) \cap \Gamma_r$.

Note that a set of the form (A) is nonempty if and only if $\kappa | a$ where κ is the greatest common divisor of the integers k_i. We refer to a finite intersection of sets of the form (A) as an *affine subset* of Γ_r^n provided that it is nonempty. Affine subsets of Γ_r are singletons, and if Y is an affine subset of Γ_r^n with $n > 1$, then there is a projection $\pi : \Gamma_r^n \to \Gamma_r^d$ with $d < n$ such that $\pi|_Y$ is injective. Also for an affine subset Y, $\vec{b} \in \mathbb{Z}^n$, and $N > 0$, the set $(h(\vec{b})Y)^{1/N}$ is either empty or an affine subset.

Suppose that Y is a set of the form (C) or (D), $\vec{b} \in \mathbb{Z}^n$, and let $N \in \mathbb{N}_+$ be a multiple of m. Then the set $(h(\vec{b})Y)^{1/N}$ is nonempty if and only if $m | a - \vec{k}\vec{b}$, and in that case

$$(h(\vec{b})Y)^{1/N} = \Big\{\vec{y} : (\vec{y}^{\vec{k}})^{N/m} \in h(\frac{\vec{k}\vec{b} - a}{m})I\Big\},$$

where I is one of P_r or Q_r. As a result, it is an open subset of Γ_r^n.

Putting these together we have the following result.

Proposition 5.4. *Let X be definable in $\mathfrak{G}$. Then there is $N \in \mathbb{N}_+$ such that for every $\vec{b} \in \mathbb{Z}^n$ the set $(h(\vec{b})X)^{1/N}$ is a union of an open set and finitely many subsets of affine sets.*

Proof. Write X as in $(*)$, and let N be the lowest common multiple of the integers m appearing in the formulas defining the sets X_{ij}. Given $i \in \{1, \ldots, s\}$, if one of the sets X_{ij} is of the form (A), then the set

$$(h(\vec{b}) \bigcap_j X_{ij})^{1/N} = \bigcap_j (h(\vec{b}) X_{ij})^{1/N}$$

is contained in an affine set. Otherwise this set is open as noted above. $\square$

Now we focus on unary definable sets with the aim of proving Theorem 1.2. Combined with Proposition 2.2, Proposition 5.4 gives the following for unary subsets of Γ_r.

Corollary 5.5. *Suppose that $X \subseteq \Gamma_r$ is definable in $\mathfrak{G}$. Then there is $N \in \mathbb{N}_+$ such that for every $n \in \{0, 1, \ldots, N-1\}$, the set $(h(n)X)^{1/N}$ is a finite union of convex sets and singletons. Moreover, the end points of the convex sets are in $\langle \Gamma_r \rangle$.*

Proof. Affine subsets of Γ_r are singletons and the open set appearing in $(h(n)X)^{1/N}$ is a certain positive Boolean combination of sets of the form:

$$\Big\{ y \in \Gamma_r : y^{kN} \neq h(-n) \Big\}.$$

and

$$\Big\{ y \in \Gamma_r : y^{kN/m} \in h(\frac{kb-n}{m}) I \Big\}$$

where I is one of P_r or Q_r.
Using Proposition 2.2, if such a combination is not empty, then it is a finite union of convex sets whose end points are in $\langle \Gamma_r \rangle$. □

Recall that a subset X of $\mathbb{Z}$ has the uniform gaps property if there is $M \in \mathbb{N}_+$ such that

$$X \cap \{x+1, \ldots, x+M\} \neq \emptyset$$

for every $x \in X$. Clearly, $\emptyset$ has the uniform gaps property and it is the only finite set that has the uniform gaps property. The following is also clear.

Lemma 5.6. *If $X, Y \subseteq \mathbb{Z}$ have the uniform gaps property. Then $X \cup Y$ has the uniform gaps property.*

Lemma 5.7. *If $C \subseteq \Gamma_r$ is a convex set, then $X = h^{-1}(C)$ has the uniform gaps property.*

Proof. Let $C = (\alpha, \beta) \cap \Gamma_r$ and let $\gamma := l(C)$. Take $m, n \in \mathbb{N}_+$ such that $h(m) < \gamma^{1/2}$ and $h(-n) < \gamma^{1/2}$. We claim that $M = \max\{m, n\}$ witnesses that X has the uniform gaps property.

Suppose that $x \in X$. If $h(x) < \alpha\gamma^{1/2}$, then $h(x)h(m) \in C$ and hence $x + m \in X$, and if $\alpha\gamma^{1/2} < h(x)$, then $h(x)h(n) \in C$ and hence $x + n \in X$. □

Remark. This proof does not work for larger models of T_r^*, because the length of C might be infinitesimal with respect to Γ_r and hence we cannot find suitable m and n.

The following is Theorem 1.2 from the Introduction.

Theorem 5.8. *Every infinite subset of $\mathbb{Z}$ definable in $\mathfrak{Z}$ has the uniform gaps property.*

Proof. Let $X \subseteq \mathbb{Z}$ be definable in $\mathfrak{Z}$ and let $Y := h(X)$.

By Corollary 5.5, there is $N > 0$ such that $(h(n)Y)^{1/N}$ is a finite union of convex sets and singletons for each $n \in \{0, 1, \dots, N-1\}$.

Using the second and the last part of Lemma 5.1

$$Y = \bigcup_{n=0}^{N-1} h(-n)((h(n)Y)^{1/N})^{(N)}.$$

Therefore by Lemma 5.6, it suffices to show that $h^{-1}\big(((h(n)Y)^{1/N})^{(N)}\big)$ has the uniform gaps property for each n. Since $y \in ((h(n)Y)^{1/N})^{(N)}$ if and only if $y = z^N$ for some $z \in (h(n)Y)^{1/N}$, it suffices to show that $h^{-1}\big((h(n)Y)^{1/N}\big)$ has the uniform gaps property. It is indeed the case using Lemmas 5.6 and 5.7. $\square$

Corollary 5.9. *Let X be an infinite subset of $\mathbb{Z}$ that is definable in $\mathfrak{Z}$. Then there is $M \in \mathbb{N}_+$ such that for every $x \in X$, the intersection $X \cap \{x-1, x-2, \dots, x-M\}$ is nonempty.*

Proof. The set $-X := \{y \in \mathbb{Z} : -y \in X\}$ is also definable in $\mathfrak{Z}$ and hence has the uniform gaps property. So there is $M \in \mathbb{N}_+$ such that given $x \in X$, we have

$$-X \cap \{-x+1, \dots, -x+M\} \neq \emptyset.$$

Say $y = -x + i \in -X$ where $1 \leq i \leq M$. Then $-y = x - i \in X$, finishing the proof. $\square$

Corollary 5.10. *Ordering of $\mathbb{Z}$ is not definable in $\mathfrak{Z}$.*

Proof. If the ordering were definable in $\mathfrak{Z}$, then so would be the set of positive elements. However, $\mathbb{N}_+$ does not satisfy the conclusion of Corollary 5.9. $\square$

Corollary 5.11. *Multiplication is not definable in $\mathfrak{Z}$.*

Proof. If multiplication on $\mathbb{Z}$ is definable, then the ordering of $\mathbb{Z}$ is also definable using Lagrange's four-square theorem. $\square$

This result follows also from the work of Bélair-Point: Corollary 4.3 of [2].

We need the following lemma to show that $\mathfrak{Z}$ is unstable.

Lemma 5.12. *Let $n > 0$, then there is $m > 0$ such that $m, 2m, \dots, nm \in P_r$ and $-m, -2m, \dots, -nm \notin P_r$.*

Proof. Let $\alpha = \min\{h(1), h(-1)\}$ and $\beta = \alpha^{\frac{1}{n+1}}$. By regular density of Γ_r take $m \in \mathbb{N}_+$ such that $\beta^{-1} < h(m)$. Clearly, this m satisfies the conclusion of the lemma. $\square$

Proposition 5.13. *The theory T_r^* is not stable.*

Proof. Let $\phi(x;y)$ be the L_P-formula $y - x \in P$. We show that this formula is unstable. For this, it suffices to prove the following: For every $n > 0$, there are $a_1, \ldots, a_n \in \mathbb{Z}$ such that

$$\mathfrak{Z} \models \phi(a_i, a_j) \Longleftrightarrow i < j.$$

So let $n > 0$ be given. Take $a_i = im$ where $m = m(n)$ is as in the lemma above. Now $a_j - a_i = (j-i)m$ and hence $\phi(a_i, a_j)$ holds in $\mathfrak{Z}$ if and only if $i < j$.

□

(We would like to thank Haydar Göral for the idea of this proof.)

6. No Reducts

We prove that there are no intermediate structures between $(\mathbb{Z}, +)$ and $(\mathbb{Z}, +, P_r)$; actually we prove the same result for $(\Gamma_r, \cdot)$ and $(\Gamma_r, \cdot, P_r)$.

We first consider a subset X of Γ_r that is definable in $\mathfrak{G}$.

Proposition 6.1. *Let $X \subseteq \Gamma_r$ be definable in $\mathfrak{G}$. Suppose that X is not definable in $(\Gamma_r, \cdot)$. Then there is $N > 0$ and $L \in \{0, 1, \ldots, N-1\}$ such that both the set $(h(L)X)^{1/N}$ and its complement contain a convex set.*

Proof. By Corollary 5.5, there is $N > 0$ such that for each n, the set $(h(n)X)^{1/N}$ is a finite union of convex sets with end points in $\langle \Gamma_r \rangle$, and singletons. Write

$$X = \bigcup_{n=0}^{N-1} h(-n)(h(n)X \cap \Gamma_r^{(N)}).$$

For each n, raising to power N is a bijection between the sets $(h(n)X)^{1/N}$ and $h(n)X \cap \Gamma_r^{(N)}$. Since X is not definable in $(\Gamma_r, \cdot)$, there is $L \in \{0, \ldots, N-1\}$ such that $(h(L)X)^{1/N}$ is infinite. If $(h(L)X)^{1/N}$ is cofinite in Γ_r, then so is $h(L)X \cap \Gamma_r^{(N)}$. Once again, since X is not definable in $(\Gamma_r, \cdot)$, $h(L)X \cap \Gamma_r^{(N)}$ cannot be cofinite in Γ_r. Therefore both $(h(L)X)^{1/N}$ and its complement contains a convex set. □

So if N and L are as in this proposition, both $(h(L)X)^{1/N}$ and its complement are finite unions of convex sets and singletons, but neither is finite. Moreover, by Corollary 5.5, the end points of the convex sets appearing in either union are in $\langle \Gamma_r \rangle$. Therefore there is a finite union of convex sets with end points in $\langle \Gamma_r \rangle$ that is definable in $(\Gamma_r, \cdot, X)$ and is not cofinite in Γ_r.

Lemma 6.2. *Let $X = (\alpha, \beta) \cap \Gamma_r^{(m)}$ and suppose that $\gamma := \alpha\beta \in \Gamma_r$. Then a convex set is definable in $(\Gamma_r, \cdot, X)$. Moreover, the end points of that convex set are in $\langle \Gamma_r \cup \{\alpha, \beta\} \rangle$.*

Proof. Let $0 \leq n < m$ be such that $\gamma \in h(n)\Gamma_r^{(m)}$.

We first consider the case that $n \neq 0$. Let $k := \gcd(n, m)$, $n' = n/k$, and $m' = m/k$. So $\gamma = h(n)\gamma_0^m = (h(n')\gamma_0^{m'})^k$ for some $\gamma_0 \in \Gamma_r$.

Let $l \in \mathbb{Z}$ be such that $h(n')\gamma_0^{m'} \in (\zeta_k^l, \zeta_k^{l+1})$. Also let

$$\delta_0 \in ((\beta^{1/k})^{-1}\zeta_k^{-l}, (\beta^{1/k})^{-1}\zeta_k^{-l+1}) \cap \gamma_0^{-m'}\Gamma_r^{(m)}, \text{ and } \delta = \delta_0^k.$$

By Lemma 5.2, the set $Y := (\alpha, \beta\delta) \cap \Gamma_r^{(m)}$ is definable in $(\Gamma_r, \cdot, X)$ and using Lemma 2.3 we have that $\gamma_Y := \alpha^{1/k}(\beta\delta)^{1/k} \in h(n')\Gamma_r^{(m)}$. In particular, $\{1, \gamma_Y, \gamma_Y^2, \ldots, \gamma_Y^{m-1}\}$ is a full set of representatives of cosets of $\Gamma_r^{(m)}$ in Γ_r.

By Proposition 2.2

$$Y^{1/k} = \bigcup_{s=0}^{k-1} \zeta_k^s(\alpha^{1/k}, (\beta\delta)^{1/k}) \cap \Gamma_r^{(m')}.$$

Take $\tau_1, \tau_2 \in \Gamma_r^{(m)}$ such that $(\alpha^{1/k}, (\beta\delta)^{1/k}) \subseteq (\alpha\tau_1, \beta\tau_2)$ and $l((\alpha\tau_1, \beta\tau_2)) < \zeta_k$. The set $(\alpha\tau_1, \beta\tau_2) \cap \Gamma_r^{(m)}$ is definable in $(\Gamma_r, \cdot, X)$ by Lemma 5.2 and hence

$$Z := ((\alpha\tau_1, \beta\tau_2) \cap \Gamma_r^{(m)}) \cap Y^{1/k} = (\alpha^{1/k}, (\beta\delta)^{1/k}) \cap \Gamma_r^{(m)}.$$

is also definable in $(\Gamma_r, \cdot, X)$.

Let $f : \Gamma_r \to \Gamma_r$ be defined as $f(x) = x^{-1}\gamma_Y$. It is clear that

$$x \in (\alpha^{1/k}, (\beta\delta)^{1/k}) \iff f(x) \in (\alpha^{1/k}, (\beta\delta)^{1/k}).$$

Therefore $f^i(Z) = (\alpha^{1/k}, (\beta\delta)^{1/k}) \cap \gamma_Y^i\Gamma^{(m)}$ for $i = 0, 1, \ldots, m-1$. Hence the proper convex set

$$(\alpha^{1/k}, (\beta\delta)^{1/k}) \cap \Gamma_r = Z \cup f(Z) \cup \cdots \cup f^{m-1}(Z)$$

is definable in $(\Gamma_r, \cdot, X)$.

Now let $n = 0$ and take $k > 0$ maximum such that $\gamma \in \Gamma_r^{(m^k)}$; say $\gamma = \gamma_0^{m^k}$ with $\gamma_0 \in (\zeta_{m^k}^l, \zeta_{m^k}^{l+1}) \cap \Gamma_r$.

Let

$$\delta_0 \in ((\beta^{1/m^k})^{-1}\zeta_{m^k}^{-l}, (\beta^{1/m^k})^{-1}\zeta_{m^k}^{-l+1}) \cap (\Gamma_r \setminus \gamma_0^{-1}\Gamma_r^{(m)}) \text{ and } \delta := \delta_0^{m^k}.$$

As in the previous case, we have $Y = (\alpha, \beta\delta) \cap \Gamma_r^{(m)}$ is definable in $(\Gamma_r, \cdot, X)$ and

$$Y^{1/m^k} = \bigcup_{s=0}^{m^k-1} \zeta_{m^k}^s(\alpha^{1/m^k}, (\beta\delta)^{1/m^k}) \cap \Gamma_r.$$

Again, there are $\tau_1, \tau_2 \in \Gamma^{(m)}$ such that

$$Z := ((\alpha\tau_1, \beta\tau_2) \cap \Gamma_r^{(m)}) \cap Y^{1/m^k} = (\alpha^{1/m^k}, (\beta\delta)^{1/m^k}) \cap \Gamma_r^{(m)}.$$

Now we have $\gamma^* := \alpha^{1/m^k}(\beta\delta)^{1/m^k} = \gamma_0\delta_0$ is not in $\Gamma_r^{(m)}$. Therefore using the previous case a proper convex set is definable in $(\Gamma_r, \cdot, Z)$, hence in $(\Gamma_r, \cdot, X)$.

It follows from the proof that the end points of the proper convex set are in $\langle \Gamma_r \cup \{\alpha, \beta\}\rangle$.

□

Lemma 6.3. *Let $X = (\alpha, \beta) \cap \Gamma_r$ and $m > 0$ with $l(\alpha, \beta) < \zeta_m$. Then for $0 < k \leq m$:*

$$\{x_1 x_2 \cdots x_k : x_1, x_2, \ldots, x_k \in X\} = (\alpha^k, \beta^k) \cap \Gamma_r$$

Proof. Let

$$Y_k := \{x_1 x_2 \cdots x_k : x_1, x_2, \ldots, x_k \in X\}.$$

It is clear from the assumptions that $Y_k \subseteq (\alpha^k, \beta^k) \cap \Gamma_r$. We prove equality by induction on k.

The case $k = 1$ is trivial, so let $k > 1$ and suppose that $Y_{k-1} = (\alpha^{k-1}, \beta^{k-1}) \cap \Gamma_r$. Then

$$\begin{aligned} Y_k &= \bigcup_{x_k \in X} Y_{k-1}\, x_k \\ &= \bigcup_{x_k \in X} ((\alpha^{k-1}, \beta^{k-1}) \cap \Gamma_r) x_k \\ &= \bigcup_{x_k \in X} (x_k \alpha^{k-1}, x_k \beta^{k-1}) \cap \Gamma_r. \end{aligned}$$

Clearly, the last union is a convex set. Since α^k and β^k are limit points of Y_k (in $\mathbb{S}$), we get

$$Y_k = (\alpha^k, \beta^k) \cap \Gamma_r.$$

□

Corollary 6.4. *Let $X = (\alpha, \beta) \cap \Gamma_r$, where $\alpha, \beta \in \langle \Gamma_r \rangle$. Then an interval is definable in $(\Gamma_r, \cdot, X)$.*

Proof. Let $N \in \mathbb{N}_+$ such that $\alpha^N, \beta^N \in \Gamma_r$, and take $\gamma \in \Gamma_r$ such that $l((\alpha, \beta\gamma)) < \zeta_N$. Then $(\alpha^N, \beta^N \gamma^N) \cap \Gamma_r$ is an interval and is definable in $(\Gamma_r, \cdot, X)$ by Lemma 6.3. □

Lemma 6.5. *Let X be a finite union of convex sets whose end points are in $\langle \Gamma_r \rangle$. If X is not cofinite in Γ_r, then an interval is definable in $(\Gamma_r, \cdot, X)$.*

Proof. After translating, we may assume that $1 \notin X$.

Let $X = \bigcup_{i=1}^m C_i$ where $C_i = (\alpha_i, \beta_i) \cap \Gamma_r$ with

$$\alpha_1 < \beta_1 \leq \alpha_2 < \beta_2 \leq \cdots \leq \alpha_m < \beta_m.$$

We prove the result by induction on m. If $m = 1$, then there is nothing to do. So suppose $m > 1$.

Put $\lambda_j = \beta_j\alpha_j^{-1}$ for all j, and $\delta_j = \alpha_{j+1}\beta_j^{-1}$ for $j = 1, \dots, m$, where $\alpha_{m+1} := \alpha_1$.

Below we consider three cases and in each one we construct a convex set definable in $(\Gamma_r, \cdot, X)$. Moreover the end points of those convex sets will still be in $\langle\Gamma_r\rangle$. Hence by using Corollary 6.4, we will be done.

Case 1: The set $\{\delta_1, \dots, \delta_m\}$ is not a singleton.
Let δ_{j_0} and δ_{j_1} be the two smallest elements of this set and let $\lambda = \min\{\delta_{j_1}\delta_{j_0}^{-1}, \lambda_1, \dots, \lambda_m\}$. Take $\gamma \in (\lambda^{1/2}, \lambda) \cap \Gamma_r$ and let $N > 0$ be such that

$$\gamma < \gamma^2 < \cdots < \gamma^{N-1} < \delta_{j_0} < \gamma^N.$$

Note that $\gamma^N < \delta_{j_1}$ and hence the union

$$Y := X \cup \gamma X \cup \gamma^2 X \cup \cdots \cup \gamma^N X$$

is a union of at most $m-1$ convex sets, yet it has infinite complement in Γ_r. So by induction hypothesis, there is a nonempty proper convex subset C of Γ that is definable in $(\Gamma_r, \cdot, Y)$. Since Y is definable in $(\Gamma_r, \cdot, X)$, the set C is definable in $(\Gamma_r, \cdot, X)$.

Case 2: The set $\{\delta_1, \dots, \delta_m\}$ is a singleton, but the set $\{\lambda_1, \dots, \lambda_m\}$ is not a singleton.
The set $Z = \Gamma_r \setminus X$ is a set as in Case 1. So a nonempty proper convex subset of Γ is definable in $(\Gamma_r, \cdot, Z)$, hence in $(\Gamma_r, \cdot, X)$.

Case 3: Both $\{\delta_1, \dots, \delta_m\}$ and $\{\lambda_1, \dots, \lambda_m\}$ are singletons.
Note that we need to have $\delta_j\lambda_j = \zeta_m$ for each j. Hence $\alpha_{j+1} = \zeta_m\alpha_j$ and $\beta_{j+1} = \zeta_m\beta_j$ for each $j = 1, \dots, m-1$. Therefore

$$X = \Big(\bigcup_{j=0}^{m-1} \zeta_m^j(\alpha_1, \beta_1)\Big) \cap \Gamma_r.$$

So $X^{(m)} = (\alpha_1^m, \beta_1^m) \cap \Gamma^{(m)}$. Therefore a convex set is definable in $(\Gamma_r, \cdot, X)$ by Lemma 6.2. □

Putting Proposition 6.1 and Lemma 6.5 together, we get the desired result for unary definable sets.

Proposition 6.6. *Let $X \subseteq \Gamma_r$ be definable in $\mathfrak{G}$, but not in $(\Gamma_r, \cdot)$. Then $(\Gamma_r, \cdot, X)$ is interdefinable with $\mathfrak{G}$.*

Now we handle the general case.

Theorem 6.7. *Let $X \subseteq \Gamma_r^n$ be definable in $\mathfrak{G}$, but not in $(\Gamma_r, \cdot)$. Then $(\Gamma_r, \cdot, X)$ is interdefinable with $\mathfrak{G}$.*

Proof. We proceed by induction on n. Proposition 6.6 serves as the case $n = 1$. So let's assume that $n > 1$ and that the result holds for $m < n$.

So let $X \subseteq \Gamma_r^n$ be definable in $\mathfrak{G}$, and take $N \in \mathbb{N}_+$ such that for every $\vec{a} \in \mathbb{Z}^n$, the set $(h(\vec{a})X)^{1/N}$ is a union of an open set and finitely many sets contained in affine sets. Write

$$X = \bigcup_{\vec{a} \in A} h(-\vec{a})(h(\vec{a})X \cap (\Gamma_r^{(N)})^n),$$

where A is a finite subset of $\mathbb{Z}^n$. It is easy to see that for each $\vec{a}$, the set $(h(\vec{a})X)^{1/N}$ has empty interior if and only if $h(\vec{a})X \cap (\Gamma_r^{(N)})^n$ is contained in a finite union of affine sets.

First suppose that $(h(\vec{a})X)^{1/N}$ has empty interior for each $\vec{a} \in A$. Then $(h(\vec{a})X)^{1/N}$ is a finite union of sets contained in an affine set and each of those sets is in bijection with a subset of Γ_r^d for some $d < n$ via a projection. If all of those subsets of Γ_r^d are definable in $(\Gamma_r, \cdot)$, then so is $(h(\vec{a})X)^{1/N}$. This cannot be correct for all $\vec{a}$, because X is not definable in $(\Gamma_r, \cdot)$. Hence we obtain a subset of Γ_r^d that is not definable in $(\Gamma_r, \cdot)$, but is definable in $(\Gamma_r, \cdot, X)$. So we obtain the result using the induction hypothesis.

So we assume that the set A^* of $\vec{a} \in A$ such that $(h(\vec{a})X)^{1/N}$ has nonempty interior is nonempty. Hence the set

$$X^* = X \bigcup_{\vec{a} \in A^*} h(-\vec{a})(h(\vec{a})X \cap (\Gamma^{(N)})^n)$$

is definable in $(\Gamma_r, \cdot, X)$ and $(h(\vec{a})X^*)^{1/N}$ has nonempty interior for each $\vec{a} \in A^*$. Put $Y := \Gamma_r \setminus X^*$. If $(h(\vec{a})Y)^{1/N}$ has empty interior for each $\vec{a} \in A^*$, then we may proceed as in the previous case. Therefore we may assume that $(h(\vec{a})Y)^{1/N}$ has nonempty interior for some $\vec{a} \in A^*$.

For $\vec{\gamma} \in \Gamma_r^{n-1}$, consider the sets

$$X^*_{\vec{\gamma}} := \{\delta \in \Gamma_r : (\vec{\gamma}, \delta) \in (h(\vec{a})X^*)^{1/N}\},$$

$$Y_{\vec{\gamma}} := \{\delta \in \Gamma_r : (\vec{\gamma}, \delta) \in (h(\vec{a})Y)^{1/N}\}$$

For each $\vec{\gamma}$, these sets are either empty or has nonempty interior. If there is $\vec{\gamma} \in \Gamma_r^{n-1}$ such that both $X^*_{\vec{\gamma}}$ and $Y_{\vec{\gamma}}$ have nonempty interior, then $X^*_{\vec{\gamma}}$ is not definable in $(\Gamma_r, \cdot)$, and we are done using the induction hypothesis. Otherwise for each $\vec{\gamma} \in \Gamma_r^{n-1}$, the set $X^*_{\vec{\gamma}}$ is either empty or is Γ_r. Then the set $Z := \{\vec{\gamma} : X^*_{\vec{\gamma}} = \Gamma_r\}$ is a subset of Γ_r^{n-1} definable in $(\Gamma_r, \cdot, X)$. If Z is definable in $(\Gamma_r, \cdot)$, then so are $(h(\vec{a})X^*)^{1/N}$ and X. Therefore Z is not definable in $(\Gamma_r, \cdot)$ and once again we are done by the induction hypothesis.

□

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