

A TALK ON THE MANN PROPERTY AND MANN PAIRS; MOSTLY MANN PAIRS

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1. MANN PROPERTY

Throughout Ω is a big ambient algebraically closed field (of characteristic 0). We are mostly interested in the subgroups Γ of Ω^\times ; later we shall include a subfield of Ω to the game.

Let's start by giving the main definition right away (after a couple of notations). Let $a_1, \dots, a_n \in \Omega$, $n \geq 1$, and consider the equation

$$(*) \quad a_1x_1 + \dots + a_nx_n = 1.$$

A *solution* of $(*)$ is a tuple $(s_1, \dots, s_n) \in \Omega^n$ such that $a_1s_1 + \dots + a_ns_n = 1$; such a solution is said to be *nondegenerate* if $\sum_{i \in I} a_i s_i \neq 0$ for all nonempty $I \subseteq \{1, \dots, n\}$, and is said to be in Γ if $(s_1, \dots, s_n) \in \Gamma^n$.

We say that Γ has the *Mann property* if for every $a_1, \dots, a_n \in \mathbb{Q}^\times$ the equation $(*)$ has only finitely many nondegenerate solutions in Γ . Well, actually it is not really necessary to limit ourselves to rational coefficients, although it takes some algebra to prove that. In any case, from now on I assume it.

The main examples of this property comes from number theory. The reason of the naming like this is that H. Mann proved that the (multiplicative) group of complex roots of unity \mathbb{U} has the Mann property, using very elementary techniques. Much later than that result some number theorists proved that every multiplicative subgroup of a field of characteristic 0 of finite rank has the Mann property. (Note that \mathbb{U} is of rank 0.) Another (not finite rank) example is $\exp(\mathbb{Q}^{\text{ac}})$ the exponentials of algebraic numbers (this is proved using the Lindemann's Theorem, but I'll mention that in the 'Mann pairs' case).

One important feature of the Mann property that it is equivalent to the Mordell-Lang property in the algebraically closed case. More precisely:

Proposition 1.1. *For algebraically closed K containing Γ , the following are equivalent:*

- (1) Γ has the Mann Property;
- (2) for every algebraic set $V \subseteq K^n$ its trace $V \cap \Gamma^n$ is a finite union of cosets of subgroups of Γ^n ;
- (3) for every $X \subseteq K^n$ that is definable in (K, Γ) its trace $X \cap \Gamma^n$ is definable in the group Γ .

Other features of the Mann property is that the theory of (Ω, Γ) is axiomatized by the theory of the abelian group Γ and that (Ω, Γ) is $(\kappa\text{-}, \text{super-})$ stable iff the abelian group Γ is $(\kappa\text{-}, \text{super-})$ stable.

2. MANN PAIRS

Now I will give a stronger form of the Mann property, which is uniform in coefficients. For this fix a subfield \mathbf{k} of Ω , which will provide the coefficients.

Call (\mathbf{k}, Γ) a *Mann pair* if for each $n \geq 1$ there is a finite $\Gamma(n) \subseteq \Gamma^n$ such that for all $a_1, \dots, a_n \in \mathbf{k}^\times$ all non-degenerate solutions of $(*)$ in Γ lie in $\Gamma(n)$.

We realized this property when looking for a function field analog for the Mann property, but then realized that H. Mann really proved that (\mathbb{Q}, \mathbb{U}) is a Mann pair. I am aiming to prove (in Section 4) the following result which is analogous to the ‘finite rank groups have the Mann property’-result.

Theorem 2.1. *Suppose Ω has characteristic zero, \mathbf{k} is algebraically closed, $\mathbf{k}^\times \cap \Gamma = \{1\}$, and Γ has finite rank. Then (\mathbf{k}, Γ) is a Mann pair.*

One other thing I want to prove is the following analog of Proposition 1.1.

Theorem 2.2. *The following are equivalent:*

- (1) (\mathbf{k}, Γ) is a Mann pair;
- (2) for all m, n , every subset of $\mathbf{k}^m \times \Gamma^n$ definable in $(\Omega, \mathbf{k}, \Gamma)$ is a finite union of sets $X \times Y$ with $X \subseteq \mathbf{k}^m$ definable in the field \mathbf{k} and $Y \subseteq \Gamma^n$ definable in the group Γ .

The next lemma provides some examples of Mann pairs.

Lemma 2.3. *Suppose Γ is torsion-free. Then the following are equivalent:*

- (1) for all $n \geq 1$ and $a_1, \dots, a_n \in \mathbf{k}^\times$, the equation $a_1x_1 + \dots + a_nx_n = 1$ has no non-degenerate solution in Γ that is different from $(1, \dots, 1)$;
- (2) whenever $n \geq 1$ and $\gamma_1, \dots, \gamma_n \in \Gamma$ are multiplicatively independent, then they are algebraically independent over \mathbf{k} .

If these conditions are satisfied, then (\mathbf{k}, Γ) is a Mann pair.

Since $\pi \notin \mathbb{Q}^{\text{ac}}$, the group $\exp(\mathbb{Q}^{\text{ac}}) \subseteq \mathbb{C}^\times$ is torsion-free, and by Lindemann’s theorem on exponentials, condition (2) of the lemma is satisfied with $\mathbf{k} = \mathbb{Q}^{\text{ac}} \subseteq \mathbb{C}$ and $\Gamma = \exp(\mathbb{Q}^{\text{ac}})$. Thus $(\mathbb{Q}^{\text{ac}}, \exp(\mathbb{Q}^{\text{ac}}))$ is a Mann pair.

Here is another application of the lemma, which applies for example to Hahn fields $K = \mathbf{k}(\Gamma)$.

Corollary 2.4. *Let $v : K^\times \rightarrow v(K^\times)$ be a valuation on a subfield K of Ω . Suppose \mathbf{k} is a subfield of K and Γ a subgroup of K^\times such that v is trivial on \mathbf{k} and injective on Γ . Then condition (2) of Lemma 2.3 is satisfied, and so (\mathbf{k}, Γ) is a Mann pair.*

3. MODEL THEORY OF MANN PAIRS

Let \mathcal{L} be the language of rings augmented by two distinct unary relation symbols. Let T be the \mathcal{L} -theory whose models are the structures $(\Omega, \mathbf{k}, \Gamma)$ where Ω is an algebraically closed field with a subfield \mathbf{k} , and a subgroup Γ of Ω^\times . Let $\mathcal{L}_\Sigma^{\text{f,gr}}$ be the 2-sorted language, with sorts f , gr (the *field sort* and the *group sort*), and with the following nonlogical symbols:

- constant symbols 0 and 1 of sort f ,
- a unary function symbol $-$ of sort $(\text{f}; \text{f})$,
- binary function symbols $+$ and \cdot of sort $(\text{f}, \text{f}; \text{f})$,
- a constant symbol 1 of sort gr ,
- a unary function symbol $^{-1}$ of sort $(\text{gr}; \text{gr})$
- a binary function symbol \cdot of sort $(\text{gr}, \text{gr}; \text{gr})$,
- for each $n \geq 1$, a $2n$ -ary relation symbol Σ_n of sort $(\text{f}, \dots, \text{f}, \text{gr}, \dots, \text{gr})$ (n places of sort f and n places of sort gr).

Of course, “1 used as a symbol of sort f ” is different from “1 used as a symbol of sort gr ” and likewise with the multiplication symbol. For a model $(\Omega, \mathbf{k}, \Gamma)$ of T we construe (\mathbf{k}, Γ) as an $\mathcal{L}_\Sigma^{\text{f,gr}}$ -structure by interpreting the symbols in the obvious way; in particular, each Σ_n is interpreted as the previously defined $\Sigma_n(\mathbf{k}, \Gamma) \subseteq \mathbf{k}^n \times \Gamma^n$.

In the next result we do *not* assume that (\mathbf{k}, Γ) is a Mann pair.

Lemma 3.1. *Let $(\Omega, \mathbf{k}, \Gamma) \models T$. Then every subset of $\mathbf{k}^m \times \Gamma^n$ definable in $(\Omega, \mathbf{k}, \Gamma)$ is definable in the $\mathcal{L}_\Sigma^{\text{f,gr}}$ -structure (\mathbf{k}, Γ) .*

The implication (3) \Rightarrow (2) of Theorem 2.2 is trivial and (2) \Rightarrow (1) is not that trivial but can be done easily. So this theorem will be established once we prove the part (1) \Rightarrow (3), which is the next result.

Proposition 3.2. *Suppose $(\Omega, \mathbf{k}, \Gamma) \models T$ and (\mathbf{k}, Γ) is a Mann pair. Then every subset of $\mathbf{k}^m \times \Gamma^n$ that is definable in $(\Omega, \mathbf{k}, \Gamma)$ is a finite union of sets $X \times Y$ with $X \subseteq \mathbf{k}^m$ definable in the field \mathbf{k} and $Y \subseteq \Gamma^n$ definable in the group Γ .*

Proof. As in the proof of the previous lemma we take an $|\Omega|^+$ -saturated elementary extension $(\Omega', \mathbf{k}', \Gamma')$ of $(\Omega, \mathbf{k}, \Gamma)$ and tuples $\vec{a}, \vec{b} \in (\mathbf{k}')^m$ and $\vec{\alpha}, \vec{\beta} \in (\Gamma')^n$ such that

$$\text{tp}_{\mathbf{k}'}(\vec{a}|\mathbf{k}) = \text{tp}_{\mathbf{k}'}(\vec{b}|\mathbf{k}) \text{ and } \text{tp}_{\Gamma'}(\vec{\alpha}|\Gamma) = \text{tp}_{\Gamma'}(\vec{\beta}|\Gamma).$$

By Lemma 3.1 it is enough to show that then

$$\text{tp}_{(\mathbf{k}', \Gamma')}((\vec{a}, \vec{\alpha})|(\mathbf{k}, \Gamma)) = \text{tp}_{(\mathbf{k}', \Gamma')}((\vec{b}, \vec{\beta})|(\mathbf{k}, \Gamma)).$$

The assumption on \vec{a} and \vec{b} gives an automorphism σ of \mathbf{k}' over \mathbf{k} such that $\sigma(\vec{a}) = \vec{b}$, and the assumption on $\vec{\alpha}$ and $\vec{\beta}$ gives an automorphism ϕ of Γ' over Γ such that $\phi(\vec{\alpha}) = \vec{\beta}$. It remains to show that this gives an automorphism

(σ, ϕ) of the $\mathcal{L}_{\Sigma}^{\text{f,gr}}$ -structure (\mathbf{k}', Γ') . This in turn reduces to establishing the following: Let N be a positive integer and $\vec{c}' \in (\mathbf{k}')^N$ and $\vec{\gamma}' \in (\Gamma')^N$. Then

$$(\mathbf{k}', \Gamma') \models \Sigma_N(\vec{c}', \vec{\gamma}') \iff (\mathbf{k}', \Gamma') \models \Sigma_N(\sigma(\vec{c}'), \phi(\vec{\gamma}')).$$

We prove the forward implication. (The backward implication follows in the same way.) Assume $(\mathbf{k}', \Gamma') \models \Sigma_N(\vec{c}', \vec{\gamma}')$. We have $\Sigma_N(\mathbf{k}, \Gamma) = \bigcup_{i \in I} P_i \times Q_i$ where I is finite, each $P_i \subseteq \mathbf{k}^N$ is definable in the field \mathbf{k} and each $Q_i \subseteq \Gamma^N$ is definable in the group Γ . Then $\Sigma_N(\mathbf{k}', \Gamma') = \bigcup_{i \in I} P'_i \times Q'_i$ where $P'_i \subseteq (\mathbf{k}')^N$ is defined in \mathbf{k}' by any formula with parameters from \mathbf{k} that defines P_i in the field \mathbf{k} , and $Q'_i \subseteq (\Gamma')^N$ is defined in Γ' by any formula with parameters from Γ that defines Q_i in the group Γ . Take $i \in I$ such that $\vec{c}' \in P'_i$ and $\vec{\gamma}' \in Q'_i$. It is clear that then $\sigma(\vec{c}') \in P_i$ and $\phi(\vec{\gamma}') \in Q_i$, so $(\mathbf{k}', \Gamma') \models \Sigma_N(\sigma(\vec{c}'), \phi(\vec{\gamma}'))$, as desired. \square

Proposition 3.3. *Let $(\Omega, \mathbf{k}, \Gamma) \models T$ be such that \mathbf{k} is algebraically closed and (\mathbf{k}, Γ) is a Mann pair. Then $(\Omega, \mathbf{k}, \Gamma)$ is stable, and if Γ is divisible, then $(\Omega, \mathbf{k}, \Gamma)$ is ω -stable.*

Proof. Take an infinite cardinal κ such that the abelian group Γ is κ -stable. We show that then $(\Omega, \mathbf{k}, \Gamma)$ is κ -stable. We can assume $\mathbf{k} \neq \Omega$, and that $|\Omega| = \kappa$. Take a κ^+ -saturated elementary extension $(\Omega', \mathbf{k}', \Gamma')$ of $(\Omega, \mathbf{k}, \Gamma)$.

By the proofs of Lemma 3.1 and Proposition 3.2, the type of an element of \mathbf{k}' over Ω in $(\Omega', \mathbf{k}', \Gamma')$ is determined by its type over \mathbf{k} in the field \mathbf{k}' . Likewise, the type of an element of Γ' over Ω in $(\Omega', \mathbf{k}', \Gamma')$ is determined by its type over Γ in the group Γ' .

Now let $t \in \Omega(\mathbf{k}' \cup \Gamma')^{\text{ac}}$, say $t \in \Omega(\vec{a}, \vec{\gamma})^{\text{ac}}$ with $\vec{a} \in (\mathbf{k}')^m$ and $\vec{\gamma} \in (\Gamma')^n$. Then $\text{tp}_{(\Omega', \mathbf{k}', \Gamma')} (t | \Omega)$ is determined by $\text{tp}_{\mathbf{k}'}(\vec{a} | \mathbf{k})$, $\text{tp}_{\Gamma'}(\vec{\gamma} | \Gamma)$ and the specification of a polynomial $P(X, Y, T) \in \Omega[X, Y, T]$ where $X = (X_1, \dots, X_m)$, $Y = (Y_1, \dots, Y_n)$ and T is a single indeterminate such that $P(\vec{a}, \vec{\gamma}, T) \in \Omega(\vec{a}, \vec{\gamma})[T]$ is irreducible and $P(\vec{a}, \vec{\gamma}, t) = 0$.

It is also easy to see that all elements of Ω' outside $\Omega(\mathbf{k}' \cup \Gamma')^{\text{ac}}$ realize the same type in $(\Omega', \mathbf{k}', \Gamma')$ over Ω .

Hence we have at most κ many different 1-types in $(\Omega', \mathbf{k}', \Gamma')$ over Ω . \square

Remark. One can show that if $(\Omega, \mathbf{k}, \Gamma)$ is a model of T and (\mathbf{k}, Γ) is a Mann pair, then the subsets \mathbf{k} and Γ of Ω are definable in the structure $(\Omega, \mathbf{k} \cup \Gamma)$. Using this fact, Proposition 3.3 also follows from Fact 2.1 and Theorem 4.8 in [1].

3.1. Axiomatizing $(\Omega, \mathbf{k}, \Gamma)$. Our goal here is to show that if $(\Omega, \mathbf{k}, \Gamma)$ is a model of T and (\mathbf{k}, Γ) is a Mann pair with \mathbf{k} small in Ω , then the elementary theory of $(\Omega, \mathbf{k}, \Gamma)$ is completely determined by the elementary theories of the field \mathbf{k} and of the group Γ . We achieve this after adding names for enough elements of \mathbf{k} and Γ to witness that (\mathbf{k}, Γ) is a Mann pair.

Fix a model $(\Omega_0, \mathbf{k}_0, \Gamma_0)$ of T such that (\mathbf{k}_0, Γ_0) is a Mann pair. For each $n \geq 2$, fix a finite subset $\Gamma_0(n)$ of Γ_0^n such that

$$\Sigma_n^{\text{nd}}(\mathbf{k}_0, \Gamma_0) = \bigcup_{\vec{\gamma} \in \Gamma_0(n)} \Sigma_n^{\text{nd}}(\mathbf{k}_0, \Gamma_0; \vec{\gamma}) \times \Gamma_0 \vec{\gamma},$$

and for each $\vec{\gamma} \in \Gamma_0(n)$, fix a basis $B(\vec{\gamma}) \subseteq \mathbf{k}_0^n$ of the \mathbf{k}_0 -linear subspace $\Sigma_n(\mathbf{k}_0, \Gamma_0; \vec{\gamma})$ of \mathbf{k}_0^n . Let $\mathcal{L}(\mathbf{k}_0, \Gamma_0)$ be the language \mathcal{L} augmented by names for the elements of $\mathbf{k}_0 \cup \Gamma_0$.

In the following definition and subsequent remarks $(\Omega, \mathbf{k}, \Gamma)$ ranges over models of T that contain $(\Omega_0, \mathbf{k}_0, \Gamma_0)$ as a substructure. We construe such $(\Omega, \mathbf{k}, \Gamma)$ as an $\mathcal{L}(\mathbf{k}_0, \Gamma_0)$ -structure in the obvious way, and say that $(\Omega, \mathbf{k}, \Gamma)$ satisfies the Mann-axioms of $(\Omega_0, \mathbf{k}_0, \Gamma_0)$ if for each $n \geq 2$:

- (1) $\Sigma_n^{\text{nd}}(\mathbf{k}, \Gamma) = \bigcup_{\vec{\gamma} \in \Gamma_0(n)} \Sigma_n^{\text{nd}}(\mathbf{k}, \Gamma; \vec{\gamma}) \times \Gamma \vec{\gamma}$;
- (2) for each $\vec{\gamma} \in \Gamma_0(n)$, the \mathbf{k} -linear subspace $\Sigma_n(\mathbf{k}, \Gamma; \vec{\gamma})$ of \mathbf{k}^n is generated by $B(\vec{\gamma})$;

Remarks. The reason for this terminology is that we have a set $\text{Mann}(\mathbf{k}_0, \Gamma_0)$ of sentences in the language $\mathcal{L}(\mathbf{k}_0, \Gamma_0)$ such that for all $(\Omega, \mathbf{k}, \Gamma)$ as above,

$$(\Omega, \mathbf{k}, \Gamma) \models \text{Mann}(\mathbf{k}_0, \Gamma_0)$$

$$\iff$$

$$(\Omega, \mathbf{k}, \Gamma) \text{ satisfies the Mann axioms of } (\Omega_0, \mathbf{k}_0, \Gamma_0).$$

In particular, if $(\Omega, \mathbf{k}, \Gamma)$ is an elementary extension of $(\Omega_0, \mathbf{k}_0, \Gamma_0)$, then $(\Omega, \mathbf{k}, \Gamma)$ satisfies the Mann axioms of $(\Omega_0, \mathbf{k}_0, \Gamma_0)$.

Theorem 3.4. *Let $(\Omega_1, \mathbf{k}_1, \Gamma_1)$ and $(\Omega_2, \mathbf{k}_2, \Gamma_2)$ be models of T such that*

- (1) $[\Omega_1 : \mathbf{k}_1] > 2$ and $[\Omega_2 : \mathbf{k}_2] > 2$;
- (2) $(\Omega_0, \mathbf{k}_0, \Gamma_0) \subseteq (\Omega_1, \mathbf{k}_1, \Gamma_1)$ and $(\Omega_0, \mathbf{k}_0, \Gamma_0) \subseteq (\Omega_2, \mathbf{k}_2, \Gamma_2)$;
- (3) $(\Omega_1, \mathbf{k}_1, \Gamma_1)$ and $(\Omega_2, \mathbf{k}_2, \Gamma_2)$ satisfy the Mann-axioms of $(\Omega_0, \mathbf{k}_0, \Gamma_0)$.

Then: $(\Omega_1, \mathbf{k}_1, \Gamma_1) \equiv_{\mathbf{k}_0 \cup \Gamma_0} (\Omega_2, \mathbf{k}_2, \Gamma_2) \iff \mathbf{k}_1 \equiv_{\mathbf{k}_0} \mathbf{k}_2$ and $\Gamma_1 \equiv_{\Gamma_0} \Gamma_2$.

For algebraically closed \mathbf{k}_i this result takes the following form:

Corollary 3.5. *Let $(\Omega_1, \mathbf{k}_1, \Gamma_1)$ and $(\Omega_2, \mathbf{k}_2, \Gamma_2)$ be models of T such that*

- (1) \mathbf{k}_1 and \mathbf{k}_2 are algebraically closed, $\mathbf{k}_1 \neq \Omega_1$, $\mathbf{k}_2 \neq \Omega_2$;
- (2) $(\Omega_0, \mathbf{k}_0, \Gamma_0) \subseteq (\Omega_1, \mathbf{k}_1, \Gamma_1)$ and $(\Omega_0, \mathbf{k}_0, \Gamma_0) \subseteq (\Omega_2, \mathbf{k}_2, \Gamma_2)$;
- (3) $(\Omega_1, \mathbf{k}_1, \Gamma_1)$ and $(\Omega_2, \mathbf{k}_2, \Gamma_2)$ satisfy the Mann axioms of $(\Omega_0, \mathbf{k}_0, \Gamma_0)$.

Then: $(\Omega_1, \mathbf{k}_1, \Gamma_1) \equiv_{\mathbf{k}_0 \cup \Gamma_0} (\Omega_2, \mathbf{k}_2, \Gamma_2) \iff \Gamma_1 \equiv_{\Gamma_0} \Gamma_2$.

4. MANN PAIRS IN FUNCTION FIELDS OF ONE VARIABLE

Let \mathbf{k} be algebraically closed, and let $F \subseteq \Omega$ be a function field of one variable over \mathbf{k} , that is, F is a field extension of finite degree of $\mathbf{k}(t)$ for some $t \in \Omega \setminus \mathbf{k}$; in particular, F has transcendence degree 1 over \mathbf{k} . Below we use some standard facts about such function fields; for proofs of these facts, see Chapter I, §2 of [3].

Let $\mathcal{R}(F|\mathbf{k})$, the *Riemann space* of F over \mathbf{k} , be the set of all valuations $v : F^\times \rightarrow \mathbb{Z}$ on F with value group $v(F^\times) = \mathbb{Z}$ that are trivial on \mathbf{k} . We let v range over $\mathcal{R}(F|\mathbf{k})$. For each $f \in F^\times$ we have $v(f) \neq 0$ for only finitely many v , and $\sum_v v(f) = 0$. Let $\mathcal{D}(F|\mathbf{k})$ be the group of *divisors* of F over \mathbf{k} , that is,

$$\mathcal{D}(F|\mathbf{k}) := \bigoplus_v \mathbb{Z}v$$

is the free abelian group on the Riemann space of F over \mathbf{k} . To $f \in F^\times$ we assign its *principal divisor* $(f) := \sum_v v(f)v \in \mathcal{D}(F|\mathbf{k})$. The group morphism

$$F^\times \rightarrow \mathcal{D}(F|\mathbf{k}), \quad f \mapsto (f)$$

has kernel \mathbf{k}^\times . In particular, if Γ is a subgroup of F^\times with $\mathbf{k}^\times \cap \Gamma = \{1\}$, then this morphism is injective on Γ , and so the image of Γ under this morphism is an isomorphic copy of Γ inside the free abelian group $\mathcal{D}(F|\mathbf{k})$. It follows that each such Γ is free as an abelian group.

Given finite $S \subseteq \mathcal{R}(F|\mathbf{k})$, an *S-unit* is an element $u \in F^\times$ such that $v(u) = 0$ for all $v \notin S$.

Lemma 4.1. *Let L be a finite-dimensional \mathbf{k} -linear subspace of F and let $\Gamma \subseteq F^\times$ be finitely generated with $\mathbf{k}^\times \cap \Gamma = \{1\}$. Then $L \cap \Gamma$ is finite.*

Proof. Let b_1, \dots, b_m be a basis of the \mathbf{k} -linear space L , and let $\gamma_1, \dots, \gamma_n$ generate the group Γ . Take a finite $S \subseteq \mathcal{R}(F|\mathbf{k})$ such that all b_i and γ_j are S -units. Take a natural number d such that $v(b_i) \geq -d$ for all $v \in S$ and $i = 1, \dots, m$. Then $v(f) \geq 0$ for all $f \in L$ and v outside S , and $v(f) \geq -d$ for all $f \in L$ and $v \in S$.

Suppose now that $\gamma \in L \cap \Gamma$. Then $v(\gamma) = 0$ for v outside S and $v(\gamma) \geq -d$ for $v \in S$. In view of $\sum_v v(\gamma) = 0$, this gives $v(\gamma) \leq |S|d$ for all $v \in S$. It follows that the image of $L \cap \Gamma$ in $\mathcal{D}(F|\mathbf{k})$ is finite, and thus $L \cap \Gamma$ is finite. \square

Let $u_1, \dots, u_n \in F$ not be all zero. We define their *height* by

$$H(u_1, \dots, u_n) := - \sum_v \min\{v(u_1), \dots, v(u_n)\}.$$

This height is projective: $H(fu_1, \dots, fu_n) = H(u_1, \dots, u_n)$ for $f \in F^\times$.

Example. Let $F = \mathbf{k}(t)$ with t transcendental over \mathbf{k} . Suppose that the polynomials $u_1, \dots, u_n \in \mathbf{k}[t]$ have no common zero in \mathbf{k} , $n \geq 1$. It is easy to check that then $H(u_1, \dots, u_n) = \max\{\deg_t u_1, \dots, \deg_t u_n\}$.

The following important bound is from [2]:

Let Ω have characteristic zero, let g be the genus of the function field $F|\mathbf{k}$, and let S be a finite subset of $\mathcal{R}(F|\mathbf{k})$ and $n \geq 2$. Suppose u_1, \dots, u_n are S -units and (u_1, \dots, u_n) is a non-degenerate solution of $x_1 + \dots + x_n = 0$. Then

$$H(u_1, \dots, u_n) \leq \frac{1}{2}(n-1)(n-2)\{|S| + \max(0, 2g-2)\}.$$

In combination with the previous lemma this has the following consequence:

Corollary 4.2. *Suppose Ω has characteristic zero and $\Gamma \subseteq F^\times$ is finitely generated with $\mathbf{k}^\times \cap \Gamma = \{1\}$. Then (\mathbf{k}, Γ) is a Mann pair.*

Proof. Take a finite $S \subseteq \mathcal{R}(F|\mathbf{k})$ such that all $\gamma \in \Gamma$ are S -units. Let $n \geq 2$ and let $a_1, \dots, a_n \in \mathbf{k}^\times$ and $\gamma_1, \dots, \gamma_n \in \Gamma$ be such that $(\gamma_1, \dots, \gamma_n)$ is a non-degenerate solution of $a_1x_1 + \dots + a_nx_n = 0$. Dividing by γ_n we arrange $\gamma_n = 1$, and we need to show that this leaves only finitely many possibilities for $(\gamma_1, \dots, \gamma_{n-1})$. Now by the bound above we have $v(\gamma_i) \geq -N$ for all $v \in S$ and $i = 1, \dots, n-1$, where $N := \frac{1}{2}(n-1)(n-2)\{|S| + \max(0, 2g-2)\}$, so each γ_i lies in the \mathbf{k} -linear subspace

$$L := \{f \in F : v(f) \geq 0 \text{ for all } v \notin S, v(f) \geq -N \text{ for all } v \in S\}$$

of F . Now L is finite-dimensional by [3], p. 7. In view of Lemma 4.1 this gives the desired finiteness. \square

Example. To illustrate the effective nature of this proof, assume that Ω has characteristic zero, and consider the case $F = \mathbf{k}(t)$ of a *rational* function field (so $g = 0$), where Γ is generated as a group by

$$t - c_1, \dots, t - c_M, \quad \text{with distinct } c_1, \dots, c_M \in \mathbf{k}.$$

Let $n \geq 2$ be given. Call a tuple $(\gamma_1, \dots, \gamma_n) \in \Gamma^n$ *reduced* if all γ_i lie in $\mathbf{k}[t]$ and $\gamma_1, \dots, \gamma_n$ have no common zero in \mathbf{k} when viewed as polynomials in t over \mathbf{k} . Let $a_1, \dots, a_n \in \mathbf{k}^\times$. Any non-degenerate solution in Γ of the equation

$$a_1x_1 + \dots + a_nx_n = 0$$

can be multiplied by an element of Γ to give a non-degenerate reduced solution $(\gamma_1, \dots, \gamma_n)$. In this situation the inequality from [2] yields

$$\max\{\deg_t \gamma_1, \dots, \deg_t \gamma_n\} \leq \frac{1}{2}(n-1)(n-2)(M+1),$$

which is satisfied by only finitely many reduced tuples $(\gamma_1, \dots, \gamma_n)$, which we can list explicitly. Given a reduced tuple $(\gamma_1, \dots, \gamma_n)$ satisfying the inequality, the existence of $a_1, \dots, a_n \in \mathbf{k}^\times$ such that $(\gamma_1, \dots, \gamma_n)$ is a non-degenerate solution of $a_1x_1 + \dots + a_nx_n = 0$ is equivalent (effectively) to the existence of a solution to a certain finite system of linear equations and inequations with coefficients in the field $\mathbb{Q}(c_1, \dots, c_M)$.

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