THE REAL FIELD WITH TWO SMALL MULTIPLICATIVE GROUPS

AYHAN GÜNAYDIN

1. Introduction

Let $\mathbb{R}$ be the *field* of real numbers. A complete axiomatization of the theory of $(\mathbb{R}, 2^{\mathbb{Z}})$, the real field with the multiplicative group $2^{\mathbb{Z}}$ as a distinguished subset, is given in [2]. That paper also determined the definable relations in this structure. The main tool was some elementary valuation theory.

The motivation for the present work is to solve an open problem from [2], namely to achieve the same things for $(\mathbb{R}, 2^{\mathbb{Z}}, 3^{\mathbb{Z}})$ and similar structures, where the method of [2] fails. Structures of this type have come up naturally in recent work by Miller and Speissegger [6], although there the real field is also equipped with further *analytic* structure, which leads to a negative result, see ... below.

A new way of approaching such questions is due to Zilber [9] , [10], where he studies the real and complex field equipped with the group of complex roots of unity and points out the relevance of a theorem of Mann on sums of roots of unity. This led to [3], where problems of the type above were solved for structures $(\mathbb{R}, G)$ with G any dense (multiplicative) subgroup of $\mathbb{R}^{>0}$ of finite rank, for example $G = 2^{\mathbb{Z}}3^{\mathbb{Z}}$. However, as shown in a stronger form in ... below, the sets $2^{\mathbb{Z}}$ and $3^{\mathbb{Z}}$ are not definable in $(\mathbb{R}, 2^{\mathbb{Z}}3^{\mathbb{Z}})$ and thus [3] does not by itself yield anything of interest concerning $(\mathbb{R}, 2^{\mathbb{Z}}, 3^{\mathbb{Z}})$.

A key fact about subgroups of $\mathbb{R}^{>0}$ of finite rank is that they have the *Mann Property*, as it is called in [3], where this property is studied in an axiomatic setting (not restricted to the real field). Special to the real setting is that groups like $2^{\mathbb{Z}}3^{\mathbb{Z}}$ are dense in $\mathbb{R}^{>0}$, and are therefore regularly dense ordered abelian groups in the sense of [7] with the order induced from $\mathbb{R}$. These two properties are of first-order nature and are instrumental in the model-theoretic analysis of [3].

In the current paper, we elaborate the techniques of [3] to determine the elementary theory of any structure $(\mathbb{R}, A, G)$ where A, G are subgroups of finite rank of $\mathbb{R}^{>0}$ with $A \supseteq G$. Two main examples are $(\mathbb{R}, 2^{\mathbb{Q}}3^{\mathbb{Q}}, 2^{\mathbb{Z}}3^{\mathbb{Z}})$ and $(\mathbb{R}, 2^{\mathbb{Z}}3^{\mathbb{Z}}, 2^{\mathbb{Z}})$. As is the case in these examples, we can assume that the larger group A is dense in $\mathbb{R}^{>0}$, and the smaller group G is either dense in $\mathbb{R}^{>0}$, or discrete. We discuss these two cases separately. We first study the theory of pairs (A, G) of regularly ordered groups in Sections 3 and 4.

Before stating the results of the paper, we recall some notations and terminology from [3]. Let K be a field of characteristic 0 and G a subgroup of its multiplicative group $K^{\times}$. Consider an equation

$$(*) \qquad a_1x_1 + \cdots + a_nx_n = 1$$

with $n \geq 2$ and *nonzero* coefficients $a_1, \dots, a_n \in \mathbb{Q}$. A *solution* of $(*)$ in G is a tuple $(g_1, \dots, g_n) \in G^n$ such that

$$a_1g_1 + \cdots + a_ng_n = 1,$$

and a *nondegenerate* solution of $(*)$ in G is a solution $(g_1, \dots, g_n)$ of $(*)$ in G such that $\sum_{i\in I} a_ig_i \neq 0$ for each nonempty subset I of $\{1, \dots, n\}$. We say that G has the *Mann property* (in K) if every equation $(*)$ has only finitely many nondegenerate solutions in G.

For a multiplicatively written abelian group A and positive integer d,

$$A^{[d]} := \{a^d : a \in A\} \qquad \text{(its subgroup of } d^{\text{th}} \text{ powers)},$$

with the corresponding elementary invariant

$$[d]A := \begin{cases} |A/A^{[d]}| & \text{if } A/A^{[d]} \text{ is finite,} \\ \infty & \text{otherwise.} \end{cases}$$

Let Δ, Γ be dense subgroups of $\mathbb{R}^{>0}$ with the Mann property such that $\Gamma \subseteq \Delta$, Δ/Γ is infinite, and $[p]\Delta$ and $[p]\Gamma$ are finite for each prime p. Let $\mathcal{L}_{\mathrm{o}}(U, V, \Delta)$ be the language of ordered rings with two distinct unary predicates U, V, and a name δ' for each element δ of Δ.

For each equation

$$a_1x_1 + \cdots + a_nx_n = 1 \qquad (n \geq 2,\ a_1, \dots, a_n \in \mathbb{Q}^{\times})$$

take a finite list of its nondegenerate solutions in Δ,

$$\delta_1 = (\delta_{11}, \dots, \delta_{1n}), \dots, \delta_k = (\delta_{k1}, \dots, \delta_{kn}),$$

and let the corresponding *Mann axiom of* Δ be the sentence

$$\forall y \Big[\big(U(y) \wedge \sum_{i=1}^{n} a_iy_i = 1 \wedge \bigwedge_{I} \sum_{i\in I} a_iy_i \neq 0 \big) \longrightarrow \bigvee_{j=1}^{k} y = \delta_j \Big]$$

in the language $\mathcal{L}_{\mathrm{o}}(U, V, \Delta)$, where

$$y = (y_1, \dots, y_n) \text{ and } U(y) = U(y_1) \wedge \cdots \wedge U(y_n),$$

the conjunction $\bigwedge_I$ is over all nonempty $I \subseteq \{1, \dots, n\}$, "$\sum_{i=1}^{n} a_iy_i = 1$" and "$\sum_{i\in I} a_iy_i \neq 0$" represent certain obvious formulas in the language of rings, and $y = \delta_j$ abbreviates $y_1 = \delta_{j1} \wedge \cdots \wedge y_n = \delta_{jn}$. The *Mann axioms of* Γ are defined in a similar fashion.

We view $\big(\mathbb{R}, \Delta, \Gamma, (\delta)_{\delta\in\Delta}\big)$ as a structure for the language $\mathcal{L}_{\mathrm{o}}(U, V, \Delta)$.

Theorem 1.1. *Let K be a real closed ordered field, let A, G be dense subgroups of $K^{>0}$, and let group homomorphisms $\delta \mapsto \delta' : \Delta \to A$ and $\gamma \mapsto \gamma' : \Gamma \to G$ be given. Then $\big(K, A, G, (\delta')_{\delta\in\Delta}\big) \equiv \big(\mathbb{R}, \Delta, \Gamma, (\delta)_{\delta\in\Delta}\big)$ if and only if*

(1) *for each $\delta \in \Delta$ and each prime number p, if δ is not a p^{th} power in Δ, then δ' is not a p^{th} power in A ;*
(2) *for each $\gamma \in \Gamma$ and each prime number p, if γ is not a p^{th} power in Γ, then γ' is not a p^{th} power in G ;*
(3) *$[p]A = [p]\Delta$ and $[p]G = [p]\Gamma$ for each prime number p ;*
(4) *for all $a_1, \dots, a_n \in \mathbb{Z}$ and $\delta_1, \dots, \delta_n \in \Delta$,*

$$a_1\delta_1 + \cdots + a_n\delta_n > 0 \Longleftrightarrow a_1\delta_1' + \cdots + a_n\delta_n' > 0 \ ;$$

(5) *$\big(K, A, (\delta')_{\delta\in\Delta}\big)$ satisfies the Mann axioms of Δ.*
(6) *$\big(K, G, (\gamma')_{\gamma\in\Gamma}\big)$ satisfies the Mann axioms of Γ.*

This theorem takes care of the structure $(\mathbb{R}, 2^{\mathbb{Q}}3^{\mathbb{Q}}, 2^{\mathbb{Z}}3^{\mathbb{Z}})$. To understand the second kind of triples, e.g. $(\mathbb{R}, 2^{\mathbb{Z}}3^{\mathbb{Z}}, 2^{\mathbb{Z}})$, let Δ be as above and Γ a subgroup of Δ with a smallest element larger than 1, say γ_0, such that for each $\delta \in \Delta$, there is $\gamma \in \Gamma$ such that $\gamma \leq \delta < \gamma\gamma_0$. Then it is easy to see that Γ is regularly discrete in the sense of [7].

Theorem 1.2. *Let K be a real closed ordered field, let A be a dense subgroup of $K^{>0}$ with a subgroup G containing γ_0' as its smallest element larger than 1, and let group homomorphisms $\delta \mapsto \delta' : \Delta \to A$ and $\gamma \mapsto \gamma' : \Gamma \to G$ be given. Then $\big(K, A, G, (\delta')_{\delta\in\Delta}\big) \equiv \big(\mathbb{R}, \Delta, \Gamma, (\delta)_{\delta\in\Delta}\big)$ if and only if*

(1) *for each $\delta \in \Delta$ and each prime number p, if δ is not a p^{th} power in Δ, then δ' is not a p^{th} power in A ;*
(2) *$[p]A = [p]\Delta$ for each prime number p ;*
(3) *for all $a_1, \dots, a_n \in \mathbb{Z}$ and $\delta_1, \dots, \delta_n \in \Delta$,*

$$a_1\delta_1 + \cdots + a_n\delta_n > 0 \Longleftrightarrow a_1\delta_1' + \cdots + a_n\delta_n' > 0 \ ;$$

(4) *$\big(K, A, (\delta')_{\delta\in\Delta}\big)$ satisfies the Mann axioms of Δ.*
(5) *$\big(K, G, (\gamma')_{\gamma\in\Gamma}\big)$ satisfies the Mann axioms of Γ.*

Notations, terminology, and a fact. We let m and n, sometimes with subscripts or otherwise decorated, range over $\mathbb{N} := \{0, 1, 2, \dots\}$, the set of natural numbers; we put $\mathbb{N}_\infty := \mathbb{N} \cup \{\infty\}$. We also let p, sometimes subscripted, range over the set $\{2, 3, 5, \dots\}$ of prime numbers. For an (additive) abelian group A and $n > 0$, let $[n]A := |A/nA|$ if A/nA is finite, $[n]A := \infty$, otherwise.

Let A be a torsion-free (additive) abelian group. Then we let

$$\mathbb{Q}A = \{\frac{a}{m} : \ a \in A, \ m > 0\} \supseteq A$$

be its divisible hull. For a subgroup A' of A and a subset X of A we let $A'\langle X\rangle_A$ be the smallest pure subgroup of A containing both A' and X; its elements are the fractions

$$\frac{a' + k_1x_1 + \cdots + k_nx_n}{m} \in \mathbb{Q}A$$

with $a' \in A',\ x_1, \dots, x_n \in X,\ k_1, \dots, k_n \in \mathbb{Z},\ m > 0$, such that

$$a' + k_1x_1 + \cdots + k_nx_n \in mA.$$

For $x = (x_1, \dots, x_n) \in A^n$ we put $A'\langle x\rangle_A := A'\langle\{x_1, \dots, x_n\}\rangle_A$. When the ambient group A is clear from the context, we write $A'\langle X\rangle$ instead of $A'\langle X\rangle_A$.

We use the following result from [4]:

Fact 1. Let T be a theory in a one-sorted language L. The following conditions are equivalent:

(1) T admits quantifier elimination;

(2) suppose $\mathcal{M}, \mathcal{N}$ are models of T, $|M| \le |L|$, and $\mathcal{N}$ is $|L|^+$-saturated; then every embedding of a proper substructure of $\mathcal{M}$ in to $\mathcal{N}$ can be extended to an embedding of a strictly larger substructure of $\mathcal{M}$ into $\mathcal{N}$.

2. Some group theoretic facts

In this section A is an (additively written) abelian group. We let a, sometimes decorated with a subscript or accent, range over A.

Let $\mathcal{A}$ be a collection of subgroups of A. We say that $\mathcal{A}$ is a *lattice* if $A_1 \cap A_2$ and $A_1 + A_2$ are in $\mathcal{A}$ whenever A_1, A_2 are. We say a lattice $\mathcal{A}$ is *distributive* if for all $A_1, A_2, A_3 \in \mathcal{A}$, we have

$$(A_1 \cap A_2) + A_3 = (A_1 + A_3) \cap (A_2 + A_3).$$

If A_1, A_2, A_3 are subgroups of A, then

$$(A_1 \cap A_2) + A_3 \subseteq (A_1 + A_3) \cap (A_2 + A_3),$$

so in order to check that a lattice $\mathcal{A}$ is distributive, it suffices that

$$(A_1 \cap A_2) + A_3 \supseteq (A_1 + A_3) \cap (A_2 + A_3)$$

for all $A_1, A_2, A_3 \in \mathcal{A}$. If $\mathcal{A}$ is a distributive lattice, then by induction it follows that for all $A_1, \dots, A_k, A_{k+1} \in \mathcal{A}$,

$$\Big(\bigcap_{i=1}^{k} A_i\Big) + A_{k+1} = \bigcap_{i=1}^{k}(A_i + A_{k+1}). \tag{2.1}$$

We do not need this, but if $\mathcal{A}$ is a distributive lattice, then

$$(A_1 + A_2) \cap A_3 = (A_1 \cap A_3) + (A_2 \cap A_3)$$

for all $A_1, A_2, A_3 \in \mathcal{A}$; see Theorem 4 in Chapter XI, Section 7 of [1].

The next lemma is a well-known generalization of the Chinese Remainder Theorem.

Lemma 2.1. *Let $\mathcal{A}$ be a distributive lattice, with $A_1, \dots, A_k \in \mathcal{A}$ and $a_1, \dots, a_k \in A$. Then the following conditions are equivalent:*

(i) there is a such that $a \equiv a_i \mod A_i$ for $i = 1, \dots, k$;

(ii) for all $i, j \in \{1, \dots, k\}$, $a_i \equiv a_j \mod (A_i + A_j)$.

Proof. If there is $a \in A$ such that $a \equiv a_i \mod A_i$ for $i = 1, \dots, k$, then $a_i \equiv a_j \mod (A_i + A_j)$ for all $i, j \in \{1, \dots, k\}$. For the other direction we proceed by induction on k. There is nothing to prove in the case $k = 1$, so let $k \geq 2$, and suppose that $a_i \equiv a_j \mod (A_i + A_j)$ for all $i, j \in \{1, \dots, k\}$. Our task is to find a congruent to a_i modulo A_i for $i = 1, \dots, k$. The induction hypothesis gives a' such that $a' \equiv a_i \mod A_i$ for $i = 1, \dots, k-1$. It follows that $a' \equiv a_k \mod (A_i + A_k)$ for $i = 1, \dots, k-1$. Hence

$$a' - a_k \in \bigcap_{i=1}^{k-1} (A_i + A_k).$$

Thus by (2.1) we have $a' - a_k \in \left(\bigcap_{i=1}^{k-1} A_i\right) + A_k$. Say $a' - a_k = c + d$, where $c \in \bigcap_{i=1}^{k-1} A_i$ and $d \in A_k$.

Now let $a = a' - c$. Then $a \equiv a_i \mod A_i$ for $i = 1, \dots, k$, as desired. □

We apply the lemma to the collection $\mathcal{A}$ of subgroups of A of the form mA. We claim that

$$mA + nA = \gcd(m, n)A, \qquad mA \cap nA = \operatorname{lcm}(m, n)A.$$

This is obvious when $m = n = 0$, with $\gcd(0, 0) := 0 =: \operatorname{lcm}(0, 0)$. Assume $m \neq 0$ or $n \neq 0$ and put $\gamma := \gcd(m, n)$, so $m = m_1\gamma$ and $n = n_1\gamma$, and $xm_1 + yn_1 = 1$ with integers x, y. This easily yields the identity on the left, and the one on the right follows likewise, using $\operatorname{lcm}(m, n) = m_1 n_1 \gamma$. Hence $\mathcal{A}$ is a lattice, and $\mathcal{A}$ is distributive because the lattice of subgroups of $\mathbb{Z}$ is distributive. Thus:

Corollary 2.2. *Given $a_1, \dots, a_k$ and $m_1, \dots, m_k$, the following conditions are equivalent:*

(i) there is a such that $a \equiv a_i \mod m_iA$ for $i = 1, \dots, k$;

(ii) for all $i, j \in \{1, \dots, k\}$, $a_i \equiv a_j \mod \gcd(m_i, m_j)A$.

We now fix a subgroup G of A, and let g, sometimes subscripted or otherwise decorated, range over G. Note that a subgroup $mA + G$ of A corresponds to the subgroup $m\bar{A}$ of $\bar{A} := A/G$. Therefore the collection of subgroups of A of the form $mA + G$ is a distributive lattice, and for all $m_1, \dots, m_k$,

$$\bigcap_{i=1}^{k} (m_iA + G) = \operatorname{lcm}(m_1, \dots, m_k)A + G = \Big(\bigcap_{i=1}^{k} m_iA\Big) + G. \tag{2.2}$$

Lemma 2.3. *Let $a_1, \dots, a_k$ and $m_1, \dots, m_k$ be given. Then the following conditions are equivalent:*

(i) there exists g such that $g \equiv a_i \mod m_iA$ for $i = 1, \dots, k$;

(ii) for all $i, j \in \{1, \dots, k\}$, $a_i \equiv a_j \mod m_iA + m_jA$ and $a_i \equiv 0 \mod (m_iA + G)$.

Proof. If $g \equiv a_i \mod m_iA$ for $i = 1, \dots, k$, then obviously

$$a_i \equiv a_j \mod m_iA + m_jA, \qquad a_i \equiv 0 \mod (m_iA + G)$$

for all $i, j \in \{1, \ldots, k\}$. Conversely, let $a_i \equiv a_j \mod m_iA + m_jA$ and $a_i \equiv 0 \mod m_iA + G$ for all $i, j \in \{1, \ldots, k\}$. The previous lemma yields a such that $a \equiv a_i \mod m_iA$ for $i = 1, \ldots, k$. Hence by (2.2),

$$a \in \bigcap_{i=1}^{k}(m_iA + G) = \big(\bigcap_{i=1}^{k} m_iA\big) + G.$$

So $a = a' + g,\ a' \in \bigcap_{i=1}^{k} m_iA$. Thus $g \equiv a_i \mod m_iA$ for all i. □

In the rest of this section we assume A is *torsion-free*. Let $a_1, \ldots, a_k$ and $g_1, \ldots, g_l$ be given, and let $m_1, \ldots, m_k, n_1, \ldots, n_l > 0$ with $l \geq 2$. If

$$(\star)\qquad \big(\bigwedge_{i=1}^{k} g \equiv a_i \mod m_iA\big) \ \wedge\ \big(\bigwedge_{j=1}^{l} g \equiv g_j \mod n_jG\big),$$

then $g - g_l = n_lg'$ and

$$(\star\star)\qquad \big(\bigwedge_{i=1}^{k} n_lg' \equiv a'_i \mod m_iA\big) \ \wedge\ \big(\bigwedge_{j=1}^{l-1} n_lg' \equiv g'_j \mod n_jG\big),$$

where $a'_i = a_i - g_l$ and $g'_j = g_j - g_l$ for $i = 1, \ldots, k$ and $j = 1, \ldots, l-1$.

For $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, l-1\}$, let $\gamma_i = \gcd(m_i, n_l)$ and $\delta_j = \gcd(n_j, n_l)$ and fix integers x_i, y_i, w_j, z_j such that $x_im_i + y_in_l = \gamma_i$ and $w_jn_j + z_jn_l = \delta_j$. Then the condition $(\star\star)$ on arbitrary g' is equivalent to

$$\big(\bigwedge_{i=1}^{k} a'_i \in \gamma_iA \ \wedge\ \frac{n_l}{\gamma_i}g' \equiv \frac{a'_i}{\gamma_i} \mod \frac{m_i}{\gamma_i}A\big)$$
$$\wedge\ \big(\bigwedge_{j=1}^{l-1} g'_j \in \delta_jG \ \wedge\ \frac{n_l}{\delta_j}g' \equiv \frac{g'_j}{\delta_j} \mod \frac{n_j}{\delta_j}G\big).$$

For $i = 1, \ldots, k$ the integers y_i and m_i/γ_i are coprime, so the condition

$$a'_i \in \gamma_iA \ \wedge\ (n_l/\gamma_i)g' \equiv a'_i/\gamma_i \mod (m_i/\gamma_i)A$$

on g' is equivalent to

$$a'_i \in \gamma_iA \ \wedge\ g' \equiv y_i(a'_i/\gamma_i) \mod (m_i/\gamma_i)A.$$

Likewise, for $j = 1, \ldots, l-1$ the condition

$$g'_j \in \delta_jG \ \wedge\ (n_l/\delta_j)g' \equiv g'_j/\delta_j \mod (n_j/\delta_j)G$$

on g' is equivalent to

$$g'_j \in \delta_jG \ \wedge\ g' \equiv z_j(g'_j/\delta_j) \mod (n_j/\delta_j)G.$$

Hence the condition $(\star\star)$ on g' is equivalent to

$$\Big(\bigwedge_{i=1}^{k} a_i' \in \gamma_i A \ \wedge \ g' \equiv y_i \frac{a_i'}{\gamma_i} \mod \frac{m_i}{\gamma_i} A\Big)$$
$$\wedge \ \Big(\bigwedge_{j=1}^{l-1} g_j' \in \delta_j G \ \wedge \ g' \equiv z_j \frac{g_j'}{\delta_j} \mod \frac{n_j}{\delta_j} G\Big).$$

If g satisfies $(\star)$, then iterating this process we get $a_1^*, \dots, a_k^*$ and integers $m_1^*, \dots, m_k^*, n > 0$ depending only on

$$a_1, \dots, a_k, \ g_1, \dots, g_l, \ m_1, \dots, m_k, \ n_1, \dots, n_l$$

and a g^* depending *also* on g such that

$$(\star\star\star) \qquad \Big(\bigwedge_{i=1}^{k} g^* \equiv a_i^* \mod m_i^* A\Big) \wedge \big(g^* \equiv 0 \mod nG\big).$$

Moreover, this process can be reversed to give g as in $(\star)$ from g^* as in $(\star\star\star)$. Therefore, by Lemma 2.3, we get

Lemma 2.4. *Let $a_1, \dots, a_k, \ g_1, \dots, g_l,$ and $m_1, \dots m_k, n_1, \dots, n_l > 0$ be given, with $l \geq 2$. Then there are $a_1^*, \dots, a_k^*$ and $m_1^*, \dots, m_k^*, n > 0$ such that the following are equivalent :*

(i) there is g satisfying $(\star)$,

(ii) for all $i, j \in \{1, \dots, k\}$, $a_i^* \equiv a_j^* \mod m_i^* A + m_j^* A$ and $a_i^* \equiv 0$ $\mod m_i^* A + nG$.

Remark. Assume also that $a_1, \dots, a_k$ lie in a pure subgroup A' of A, and $g_1, \dots, g_l \in G' := G \cap A'$. Then G' is pure in G, and the proof shows that the conclusion of the lemma holds with $a_1^*, \dots, a_k^* \in A'$ and $m_1^*, \dots, m_k^*, n$ depending only on

$$(A', G', a_1, \dots, a_k, m_1, \dots, m_k, g_1, \dots, g_l, n_1, \dots, n_l),$$

not on the ambient structure (A, G). The purity of A' in A also shows that in (ii) one can replace A and G by A' and G'. It follows that if there is a g as in (i), then there is already such a g in G'.

For each p, let $[p]A = p^{e(p)}$ and $[p]G = p^{f(p)}$ with $e(p), f(p) \in \mathbb{N}_\infty$. Let $e := \big(e(p)\big)$ and $f := \big(f(p)\big)$ and call e, f the *system of prime invariants of* (A, G). Note that if $a_1, \dots, a_n \in A$ are $\mathbb{Z}$-linearly dependent, then $a_1/pA, \dots, a_n/pA$ are linearly dependent in the $\mathbb{F}_p$-linear space A/pA. In particular, if A has finite rank, then for every p,

$$e(p) = \dim_{\mathbb{F}_p} A/pA \leq \operatorname{rk} A < \infty.$$

Let $\mathcal{L}_{\text{ab}} := \{+, -, 0\}$ be the language of abelian groups, and let $\mathcal{L}_{\text{ab}}(V)$ extend $\mathcal{L}_{\text{ab}}$ by a unary predicate symbol V. We consider (A, G) as an $\mathcal{L}_{\text{ab}}(V)$-structure. Note that if (B, H) is an $\mathcal{L}_{\text{ab}}(V)$-structure and $(A, G) \equiv (B, H)$, then B is also a torsion-free abelian group with subgroup H, and (A, G) and (B, H) have the same system of prime invariants.

In the rest of this section we assume that A/pA and G/pG are finite for each p. Hence for $n > 0$, the group A/nA is finite, and

$$[n]A = |A/nA| = [p_1]A \cdots [p_k]A$$

if $n = p_1 \cdots p_k$. Likewise, $[n]G = |G/nG| < \infty$ for $n > 0$. Now let (A', G') be an $\mathcal{L}_{\mathrm{ab}}(V)$-substructure of (A, G) such that A' is pure in A (so $G' = G \cap A'$ is pure in G), and

$$[p]A' \geq [p]A \text{ and } [p]G' \geq [p]G,$$

for each p. These assumptions yield that for each $n > 0$ the natural group morphisms

$$A'/nA' \to A/nA, \qquad G'/nG' \to G/nG$$

are isomorphisms.

Now let B be a torsion-free abelian group with subgroup H such that (B, H) has the same system of prime invariants as (A, G). Let (B', H') be an $\mathcal{L}_{\mathrm{ab}}(V)$-substructure of (B, H) such that $|B'| = |A'|$, B' is pure in B and such that $[p]B' \geq [p]B$, and $[p]H' \geq [p]H$ for each p. Assume that (A, G) and (B, H) be κ-saturated for an infinite cardinal $\kappa > |A'| = |B'|$, and let

$$\iota : \ (A', G') \to (B', H')$$

be an $\mathcal{L}_{\mathrm{ab}}(V)$-isomorphism.

Lemma 2.5. *Let g be given. Then there is $h \in H$ such that for all $a' \in A'$, $g' \in G'$, $m, n > 0$, and $k, l \in \mathbb{Z}$,*

$$kg \equiv a' \mod mA \iff kh \equiv \iota(a') \mod mB, \tag{2.3}$$

$$lg \equiv g' \mod nG \iff lh \equiv \iota(g') \mod nH. \tag{2.4}$$

Proof. Note that if $h \in B$ satisfies (2.4) for $g' = 0$ and $l = n = 1$, then $h \in H$, since $g \in G$. So we can disregard the requirement that $h \in H$.

Claim. Let $h \in B$ be such that for all $a' \in A', \ g' \in G', \ m, n > 0$,

$$g \equiv a' \mod mA \iff h \equiv \iota(a') \mod mB, \tag{2.5}$$

$$g \equiv g' \mod nG \iff h \equiv \iota(g') \mod nH. \tag{2.6}$$

Then (2.3), and (2.4) hold for all $a' \in A', \ g' \in G', \ m, n > 0$, and $k, l \in \mathbb{Z}$.

Proof of the claim. Let $k \in \mathbb{Z}$ and $m > 0$. Let $\gamma := \gcd(k, m) = xk + ym$ with $x, y \in \mathbb{Z}$. Thus $1 = x\frac{k}{\gamma} + y\frac{m}{\gamma}$, so x and $\frac{m}{\gamma}$ are coprime, hence

$$\begin{aligned} kg - a' \in mA &\iff a' \in \gamma A \ \text{ and } \frac{k}{\gamma}g - \frac{a'}{\gamma} \in \frac{m}{\gamma}A \\ &\iff a' \in \gamma A \ \text{ and } g - \frac{xa'}{\gamma} \in \frac{m}{\gamma}A. \end{aligned}$$

Likewise, if $l \in \mathbb{Z}$, $n > 0$, and $\delta := \gcd(l, n) = wl + zn$ with $w, z \in \mathbb{Z}$, then

$$lg - g' \in nG \iff g' \in \delta G \text{ and } g - \frac{wg'}{\delta} \in \frac{n}{\delta} G.$$

So the claim follows by the purity of A' in A and G' in G.

Let $a'_1, \ldots, a'_s \in A'$, $g'_1, \ldots, g'_t \in G'$, and let $m_1, \ldots, m_s, n_1, \ldots, n_t > 0$ be such that $g \equiv a'_i \mod m_i A$ for $i = 1, \ldots, q$, $g \not\equiv a'_i \mod m_i A$ for $i = q+1, \ldots, s$, $g \equiv g'_j \mod n_j G$ for $j = 1, \ldots, r$, and $g \not\equiv g'_j \mod n_j G$ for $j = r+1, \ldots, t$. By saturation it suffices to find $b \in B$ such that $b \equiv \iota(a'_i) \mod m_i B$ for $i = 1, \ldots, q$, $b \not\equiv \iota(a'_i) \mod m_i B$ for $i = q+1, \ldots, s$, $b \equiv \iota(g'_j) \mod n_j H$ for $j = 1, \ldots, r$, and $b \not\equiv \iota(g'_j) \mod n_j H$ for $j = r+1, \ldots t$.

By the isomorphisms $A'/mA' \simeq A/mA$ and $G'/nG' \simeq G/nG$ for $m, n > 0$ we have $g \equiv a'_i + c'_i \mod m_i A$ and $g \equiv g'_j + d'_j \mod n_j G$ for $i = q+1, \ldots, s$, and $j = r+1, \ldots, t$, where $c'_{q+1}, \ldots, c'_s \in A'$, and $d'_{r+1}, \ldots, d'_t \in G'$. Note that then $c'_i \notin m_i A$ for $i = q+1, \ldots, s$ and $d'_j \notin n_j G$ for $j = r+1, \ldots, t$. Hence, if $b \in B$ and $b \equiv \iota(a'_i + c'_i) \mod m_i B$ for $i = q+1, \ldots, s$ and $b \equiv \iota(g'_j + d'_j) \mod n_j H$ for $j = r+1, \ldots, t$, then $b \not\equiv \iota(a'_i) \mod m_i B$ for $i = q+1, \ldots, s$, and $b \not\equiv \iota(g'_j) \mod n_j H$ for $j = r+1, \ldots, t$. Therefore, replacing a'_i for $i > q$ by $a'_i + c'_i$ and g'_j by $g'_j + d'_j$ for $j > r$, we reduce to the case that $q = s$ and $r = t$ with the incongruences replaced by congruences.

Lemma 2.4 and the subsequent remark show that these $s + t$ congruences satisfied by g are satisfied by some $g' \in G'$, and then $b := \iota(g') \in B'$ satisfies the corresponding congruences in (B, H), using that B' is pure in B. □

3. Pairs of regularly dense ordered abelian groups

In Section 5 we apply the results of the current section and the next section to structures $(\mathbb{R}, A, G)$ where $\mathbb{R}$ is the field of real numbers and A and $G \subseteq A$ are subgroups of the multiplicative group $\mathbb{R}^{>0}$. In this situation, if A/G is finite, then $(\mathbb{R}, A, G)$ and $(\mathbb{R}, G)$ have the same definable relations, and the structures $(\mathbb{R}, G)$ have been studied in [3]. For this reason our main interest is in the case that A/G is infinite. Later we use multiplicative notation in dealing with subgroups of multiplicative groups of fields, but in the current section additive notation is more convenient.

Let A be an ordered abelian group; so A is in particular torsion-free. We recall here some terminology from [7]. We say that A is *regularly dense* if A is non-trivial and for every p and all $a < b$ in A we have $pA \cap (a, b) \neq \emptyset$.

The reason for introducing this notion is its first-order nature and the fact that ordered subgroups of the multiplicative group $\mathbb{R}^{>0}$ that are dense in $\mathbb{R}^{>0}$ are regularly dense.

If A is regularly dense, then A has no smallest positive element, so A, as a linearly ordered set, is dense without endpoints.

Note that if A is regularly dense and G is an ordered subgroup of A and dense in A, then G is regularly dense.

In the rest of this section A is a regularly dense ordered abelian group, G is a regularly ordered subgroup of A such that G is dense in A, A/G is infinite, and $[p]A$ and $[p]G$ are finite for each p. (These assumptions are clearly satisfied if A is a dense subgroup of $\mathbb{R}^{>0}$ of finite rank and G is a subgroup of A such that G is dense in A and A/G is infinite.)

Let $\mathcal{L}_{\text{oab}} = \{<, 0, +, -\}$ be the language of ordered abelian groups, and $\mathcal{L}_{\text{oab}}(V)$ the language extending $\mathcal{L}_{\text{oab}}$ by a unary predicate symbol V.

Consider a second pair (B, H) where B is a regularly dense ordered abelian group and H is an ordered subgroup of B such that H is dense in B, and B/H is infinite. Assume also that (A, G) and (B, H) have the same system of prime invariants and are κ-saturated where κ is an uncountable cardinal. Let $\mathcal{I}$ be the set of isomorphisms $\iota : (A', G') \to (B', H')$ between $\mathcal{L}_{\text{oab}}(V)$-substructures (A', G') of (A, G) and (B', H') of (B, H) of cardinality less than κ, such that:

(1) A', B' are pure subgroups of A, B, respectively;
(2) for each p,

$$[p]A' \geq [p]A, \quad [p]G' \geq [p]G, \quad [p]B' \geq [p]B, \quad [p]H' \geq [p]H.$$

If $\iota : (A', G') \to (B', H')$ is in $\mathcal{I}$, then $G' = A' \cap G$ and $H' = B' \cap H$ are pure subgroups of G and H respectively; moreover for each p

$$[p]A' = [p]A, \quad [p]G' = [p]G, \quad [p]B' = [p]B, \quad [p]H' = [p]H.$$

We proceed to show that $\mathcal{I}$ has the back-and-forth property. So let

$$\iota : (A', G') \to (B', H')$$

be in $\mathcal{I}$, and $a \in A \setminus A'$. We want to extend ι to an $\mathcal{L}_{\text{oab}}(V)$-isomorphism in $\mathcal{I}$ whose domain contains a.

First consider the case $a \in G$. Then by Lemma 2.5 we can take $b \in H$ such that for all $a' \in A'$, $g' \in G'$, $m, n > 0$ and $k, l \in \mathbb{Z}$:

$$a' + ka \in mA \iff \iota(a') + kb \in mB,$$
$$g' + la \in nG \iff \iota(g') + lb \in nH.$$

We wish to modify b so that it still satisfies the above and such that in addition for each $N \in \mathbb{N}^{>0}$, Nb realizes the cut in B' corresponding to the cut of Na in A' via ι. By saturation, it suffices to modify b so that for given $N_1, \dots, N_s \in \mathbb{N}^{>0}$, $N_1 b, \dots, N_s b$ realize the cuts in B' corresponding to the cuts of $N_1 a, \dots, N_s a$ in A' via ι, respectively. Moreover, taking $N := N_1 \cdots N_s$, we reduce to the case that $s = 1$.

So let $N \in \mathbb{N}^{>0}$ be given. Using saturation and the density of H in B we pick $b_1 \in H$ so that b_1 realizes the cut in B' corresponding to the cut of Na in A'. Take $\varepsilon \in H$ such that $0 < \varepsilon < b'$ for all $b' \in B'$; so all elements of $[b_1, b_1 + \varepsilon] \subseteq H$ realize the same cut in B' as b_1. As H is regularly dense, saturation yields $h \in [b_1 - Nb, b_1 + \varepsilon - Nb] \cap \bigcap_{n>0} nH$. Then $Nb + h$ realizes the cut in B' corresponding to the cut of Na in A' via ι. It follows that $b + \frac{h}{N}$ is a modification of b as desired.

We can now assume that for each $N \in \mathbb{N}^{>0}$, Nb realizes the cut in B' corresponding to the cut of Na in A'. This allows us to extend ι to an $\mathcal{L}_{\text{oab}}(V)$-isomorphism

$$\big(A'\langle a\rangle_A, G'\langle a\rangle_G\big) \to \big(B'\langle b\rangle_B, H'\langle b\rangle_H\big), \qquad a \mapsto b.$$

Note that $A'\langle a\rangle_A$ and $B'\langle b\rangle_B$ are pure subgroups of A and B respectively, and that $A'\langle a\rangle_A \cap G = G'\langle a\rangle_G$ and $B'\langle b\rangle_B \cap H = H'\langle b\rangle_H$. It remains to check that (2) holds with $A'\langle a\rangle_A$, $G'\langle a\rangle_G$, $B'\langle b\rangle_B$, $H'\langle b\rangle_H$ instead of A', G', B', H'. Let any p be given. By purity we have

$$[p]A'\langle a\rangle_A \geq [p]A',\ [p]G'\langle a\rangle_G \geq [p]G',\ [p]B'\langle b\rangle_B \geq [p]B,\ [p]H'\langle b\rangle_G \geq [p]H',$$

which in view of (2) yields the desired inequalities. Therefore the above extension of ι is in $\mathcal{I}$.

Next, consider the case that $a \in A'\langle G\rangle_A$. Then $a = (a' + g)/m$, where $a' \in A'$, $g \in G$, and $m > 0$ with $a' + g \in mA$. Then by the previous case, we can extend ι to an $\mathcal{L}_{\text{oab}}(V)$-isomorphism in $\mathcal{I}$ with g in its domain, and thus a in its domain as well.

Finally, assume that $a \in A \setminus A'\langle G\rangle_A$. Since B/H is infinite, it follows by saturation that $B'\langle H\rangle_B \neq B$, and that $B \setminus B'\langle H\rangle_B$ is dense in B. As in the first case we get $b \in B \setminus B'\langle H\rangle_B$ such that for each $N \in \mathbb{N}^{>0}$, Nb realizes the cut in B' corresponding via ι to the cut that Na realizes in A', and for all $a' \in A'$, $n > 0$, and $k \in \mathbb{Z}$

$$a' + ka \in mA \Longleftrightarrow \iota(a') + kb \in mB.$$

Hence ι extends to an $\mathcal{L}_{\text{oab}}(V)$-isomorphism

$$\big(A'\langle a\rangle_A, G'\big) \to \big(B'\langle b\rangle_B, H'\big), \quad a \mapsto b.$$

By the same argument as in the first case it follows that this extension of ι is in $\mathcal{I}$. This finishes the proof of the fact that $\mathcal{I}$ has the back-and-forth property.

Corollary 3.1. *Let (A', G') be a substructure of (A, G) such that A' is pure in A, and regularly dense, and G' is dense in A'. Suppose that A'/G' is infinite and (A, G) and (A', G') have the same system of prime invariants. Then $(A', G') \preceq (A, G)$.*

Proof. Let κ be an uncountable cardinal bigger than $|A'|$, and take a κ-saturated elementary extension (A^*, G^*) of (A', G'). We may assume that (A, G) is κ-saturated as well. So there is a back-and-forth system $\mathcal{I}$ between (A, G) and (A^*, G^*), containing the identity map on (A', G'). This yields the desired result. □

Let $e = \big(e(p)\big)$, and $f = \big(f(p)\big)$ be two families of natural numbers indexed by the prime numbers. Let $T_{e,f}$ be the $\mathcal{L}_{\mathrm{oab}}(V)$-theory whose models are the pairs (A, G), where A is a regularly dense ordered abelian group, G is an ordered subgroup of A such that G is dense in A, A/G is infinite, and the system of prime invariants of (A, G) is e, f. To show that $T_{e,f}$ has a model, let $\mathbb{Z}_{(p)}$ be the additive group of the localization of $\mathbb{Z}$ at the prime ideal $p\mathbb{Z}$, viewed as a subgroup of the additive group of $\mathbb{Q}$; in other words $\mathbb{Z}_{(p)} = \{\frac{x}{y} \in \mathbb{Q} : x, y \in \mathbb{Z}, p \not| y\}$. Note that $[p]\mathbb{Z}_{(p)} = p$ and $[p']\mathbb{Z}_{(p)} = 1$ for every $p' \neq p$.

First assume that f is not identically 0. Put

$$A := \bigoplus_p \mathbb{Z}_{(p)}^{e(p)} \oplus \bigoplus_p \mathbb{Q}^{f(p)}, \text{ and } \ G := \bigoplus_p \mathbb{Z}_{(p)}^{f(p)},$$

Consider G as a subgroup of A by identifying G with a subgroup of the second summand $\bigoplus_p \mathbb{Q}^{f(p)}$ of A in the obvious way. Note that the system of prime invariants of (A, G) is e, f. Since $\mathbb{R}$ is an infinite dimensional $\mathbb{Q}$-vector space, we can embed A as an abelian group into the additive group of $\mathbb{R}$. Such an embedding makes A and G into regularly dense ordered abelian groups with the order induced by $\mathbb{R}$, and then $(A, G) \models T_{e,f}$.

Next, let $f(p) = 0$ for each p, and let e be not identically 0. Put

$$A = \mathbb{Q} \oplus \bigoplus_p \mathbb{Z}_{(p)}^{e(p)}, \text{ and}$$

$$G = \mathbb{Q}.$$

Consider G as the first summand in the above direct sum decomposition of A. As in the previous paragraph we can embed A as an abelian group into the additive group of $\mathbb{R}$; such an embedding makes A and G into regularly dense ordered abelian groups with the order induced from $\mathbb{R}$, and then $(A, G) \models T_{e,f}$.

Finally, let $e(p) = f(p) = 0$ for each p. Let α be an irrational real number, and let $A := \mathbb{Q} + \mathbb{Q}\alpha \subseteq \mathbb{R}$, and $G := \mathbb{Q} \subseteq \mathbb{R}$. Then (A, G) equipped with the order induced from $\mathbb{R}$ is a model of $T_{e,f}$.

For each n we put $\mathbf{n} := (n_p)$ with $n_p = n$ for all p.

Next we prove a quantifier eliminability result for $T_{\mathbf{0},f}$. Fix a tuple of distinct variables $x = (x_1, \dots, x_m)$. A *special* $\mathcal{L}_{\mathrm{oab}}(V)$-formula in x is a formula, $\psi(x)$, of the form $\exists y\big(V(y) \wedge \theta_V(y) \wedge \phi(x, y)\big)$, where $y = (y_1, \dots, y_n)$ is a tuple of distinct variables, $\theta(y)$ and $\phi(x, y)$ are $\mathcal{L}_{\mathrm{oab}}$-formulas, and $\theta_V(y)$ is the V-restriction of $\theta(y)$ as defined in [3].

Lemma 3.2. *Each $\mathcal{L}_{\mathrm{oab}}(V)$-formula, $\psi(x)$, is $T_{\mathbf{0},f}$-equivalent to a boolean combination of special $\mathcal{L}_{\mathrm{oab}}(V)$-formulas in x.*

Proof. Let (A, G) and (B, H) be two $\aleph_1$-saturated models of $T_{\mathbf{0},f}$, and let $\mathcal{I}$ be the back-and-forth system between (A, G) and (B, H) as constructed earlier in this subsection (taking $\kappa = \aleph_1$). Now let $a \in A^m$ and $b \in B^m$ satisfy the same special formulas in A and B respectively. It suffices to prove that a and b satisfy the same types in (A, G) and (B, H) respectively. To show this, we find an element ι of $\mathcal{I}$ with $\iota(a) = b$.

Let $\mathrm{rk}(G\langle a\rangle_A/G) = r$. We may assume that

$$\mathrm{rk}(G\langle a_1, \dots, a_r\rangle_A/G) = r, \text{ and } a_i \in G\langle a_1, \dots, a_r\rangle_A$$

for $r < i \leq m$. Then since a and b satisfy the same special formulas, we have $\mathrm{rk}(H\langle b_1, \dots, b_r\rangle_B/H) = r$ and $b_i \in H\langle b_1, \dots, b_r\rangle_B$ for $r < i \leq m$.

Now let G' be a countable pure subgroup of G such that

$$\mathrm{rk}(G'\langle a\rangle_A/G') = r, \text{ and } [p]G' = [p]G$$

for each p. Note that $G'\langle a\rangle_A$ is divisible, since it is a pure subgroup of the divisible group A. Thus $[p]G'\langle a\rangle_A = [p]A = 1$ for every p. Hence $(G'\langle a\rangle_A, G')$ is a countable substructure of (A, G), and moreover (1) and (2) are satisfied with $G'\langle a\rangle_A$ in the place of A'.

Enumerate G' as $g = (g_0, g_1, \dots)$, and let $y = (y_0, y_1, \dots)$ be a countable tuple of distinct variables. If $\theta_1(y), \dots, \theta_k(y)$ and $\phi_1(x, y), \dots, \phi_k(x, y)$ are $\mathcal{L}_{\mathrm{oab}}$-formulas such that $G \models \theta_i(g)$ and $A \models \phi_i(a, g)$ for $i = 1, \dots, k$, then

$$(A, G) \models \exists y\big(V(y) \wedge \theta_V(y) \wedge \phi(a, y)\big),$$

where $\theta(y) := \bigwedge_{i=1}^k \theta_i(y)$, and $\phi(x, y) := \bigwedge_{i=1}^k \phi_i(x, y)$. Therefore

$$(B, H) \models \exists y\big(V(y) \wedge \theta_V(y) \wedge \phi(b, y)\big).$$

So we have a partial y-type over b in (B, H), consisting of formulas $\theta_V(y)$ and $\phi(b, y)$ where $G \models \theta(g)$ and $A \models \phi(a, g)$. Thus by saturation there is a countable tuple $h = (h_0, h_1, \dots)$ realizing this type.

Now let $H' = \{h_0, h_1, \dots\}$. Then H' is a countable pure subgroup of H which is $\mathcal{L}_{\mathrm{oab}}$-isomorphic to G' such that $\mathrm{rk}(H'\langle b\rangle_B/H') = r$, $[p]H' = [p]H$ for every p, and $H'\langle b\rangle_B$ is divisible. Moreover there is an ordered group isomorphism $G'\langle a\rangle_A \to H'\langle b\rangle_B$ sending g_i to h_i for $i = 0, 1 \dots$, and a to b. Now the $\mathcal{L}_{\mathrm{oab}}(V)$-isomorphism

$$\iota : \big(G'\langle a\rangle_A, G'\big) \to \big(H'\langle b\rangle_B, H'\big)$$

is in $\mathcal{I}$, since (1) and (2) are satisfied with $G'\langle a\rangle_A, H'\langle b\rangle_B$ in the place of A', B'. □

Also as a consequence of Lemma 3.2, we get

Corollary 3.3. *The $\mathcal{L}_{\mathrm{oab}}(V)$-theory $T_{\mathbf{0},f}$ is complete.*

Proof. Let σ be an $\mathcal{L}_{\mathrm{oab}}(V)$-sentence. Then by Lemma 3.2, we may assume that σ is of the form

$$\exists y\big(V(y) \wedge \theta_V(y) \wedge \psi(y)\big),$$

where $\theta(y)$ and $\psi(y)$ are $\mathcal{L}_{\text{oab}}$-formulas. Moreover we may choose $\psi(y)$ to be quantifier-free since the theory of divisible ordered abelian groups admit quantifier elimination. Thus σ holds true in a model (A, G) of $T_{\mathbf{0},f}$ if and only if $\exists y(\theta(y) \wedge \psi(y))$ holds true in G. By [7], the theory of regularly dense ordered abelian groups whose system of prime invariants is f is complete. Thus the completeness of $T_{\mathbf{0},f}$ follows. □

The next example illustrates that the completeness result above cannot be generalized to $T_{e,f}$ for arbitrary e, f.

Example. Let $\alpha, \beta \in \mathbb{R}$ be algebraically independent over $\mathbb{Q}$. Consider two models $(\mathbb{Z} + \mathbb{Q}\alpha + \mathbb{Q}\beta, \mathbb{Z} + \mathbb{Q}\alpha)$ and $(\mathbb{Q} + \mathbb{Q}\alpha + \mathbb{Z}\beta, \mathbb{Z} + \mathbb{Q}\alpha)$ of $T_{\mathbf{1},\mathbf{1}}$. Let $m > 0$ and define σ_m to be the sentence $\forall x(V(x) \rightarrow \exists y(x = my))$. Then for each $m > 0$

$$(\mathbb{Z} + \mathbb{Q}\alpha + \mathbb{Q}\beta, \mathbb{Z} + \mathbb{Q}\alpha) \models \neg\sigma_m$$

and

$$(\mathbb{Q} + \mathbb{Q}\alpha + \mathbb{Z}\beta, \mathbb{Z} + \mathbb{Q}\alpha) \models \sigma_m.$$

Thus $T_{\mathbf{1},\mathbf{1}}$ is not complete.

For $n > 0$, a similar example can be constructed to show that $T_{\mathbf{n},\mathbf{n}}$ is not complete.

By [7], and page 75 of [3]:

Lemma 3.4. *The theory of regularly dense ordered abelian groups whose system of prime invariants is f, admits elimination of quantifiers after extending it by definitions as follows: augment $\mathcal{L}_{\text{oab}}$ by extra unary predicates D_d $(d > 0)$, with defining axioms*

$$\forall x\big(D_d(x) \leftrightarrow \exists y(x = dy)\big).$$

Proposition 3.5. *Let (A, G) be a model of $T_{\mathbf{0},f}$. Then every subset of A^m definable in (A, G) is a boolean combination of subsets of A^m defined in (A, G) by formulas $\exists y\big(V(y) \wedge \psi(x, y)\big)$, where $\psi(x, y)$ is a quantifier-free $\mathcal{L}_{\text{oab}}(A)$-formula(Here $\mathcal{L}_{\text{oab}}(A)$ is the language of ordered abelian groups augmented by names for the elements of A).*

Proof. By Lemma 3.2 it is enough to prove that the subsets of A^m defined by special formulas are of the required form. Let $\theta(y)$ and $\phi(x, y)$ be $\mathcal{L}_{\text{oab}}$-formulas. We find a quantifier-free $\mathcal{L}_{\text{oab}}(A)$-formula $\psi(x, z)$ with z a tuple of variables such that $\exists y\big(V(y) \wedge \theta_V(y) \wedge \phi(x, y)\big)$ is equivalent in (A, G) to $\exists z\big(V(z) \wedge \psi(x, z)\big)$.

By Lemma 3.4, the set $\{g \in G^n : G \models \theta(g)\}$ is a boolean combination of subsets of G^n of the form

$$\{g \in G^n : k_1 g_1 + \cdots + k_n g_n = 0\},$$
$$\{g \in G^n : k_1 g_1 + \cdots + k_n g_n < 0\},$$
$$\{g \in G^n : k_1 g_1 + \cdots + k_n g_n \in dG\},$$

where $k_1, \dots, k_n \in \mathbb{Z}$ and d is a positive integer.

Since we assume that $f(p)$ is a non-negative integer, the complement of the set

$$\{g \in G^n : k_1 g_1 + \cdots + k_n g_n \in dG\}$$

is a finite union of sets of the form

$$\{g \in G^n : k_1 g_1 + \cdots + k_n g_n \in g_0 + dG\},$$

where $g_0 \in G$.

Thus we get the desired $\psi(x, z)$, which can be chosen to be quantifier-free as the theory of divisible ordered abelian groups admits quantifier elimination. □

To get results analogous to Corollary 3.3 and Proposition 3.5 for arbitrary e, f, we augment the language $\mathcal{L}_{\mathrm{oab}}(V)$ by adding a unary predicate D_n for each $n > 0$; we call this new language $\mathcal{L}^*_{\mathrm{oab}}(V)$.

Now let $T^*_{e,f}$ be the $\mathcal{L}^*_{\mathrm{oab}}(V)$-theory extending $T_{e,f}$ by the following axiom for each $n > 0$,

$$\forall x\big(D_n(x) \leftrightarrow \exists y(x = ny)\big). \tag{3.1}$$

Proposition 3.6. *The $\mathcal{L}^*_{\mathrm{oab}}(V)$-theory $T^*_{e,f}$ is model complete.*

Proof. Let $(A', G') \subseteq (A, G)$ be two models of $T^*_{e,f}$. Note that by Lemma 3.4, we get that $A' \preceq A$. In particular A' is a pure subgroup of A. Then by using Corollary 3.1, we get $(A', G') \preceq (A, G)$. □

Let $n > 0$. Write $n = p_1^{m_1} \cdots p_d^{m_d}$, and let

$$k(n) := p_1^{m_1 e(p_1)} \cdots p_d^{m_d e(p_d)} \text{ and } l(n) := p_1^{m_1 f(p_1)} \cdots p_d^{m_d f(p_d)}.$$

To get a quantifier elimination result, we augment $\mathcal{L}^*_{\mathrm{oab}}(V)$ further by adding a unary predicate E_n, $k(n) + l(n)$ many distinct constant symbols, $c_1^{(n)}, \dots, c_{k(n)}^{(n)}, d_1^{(n)}, \dots, d_{l(n)}^{(n)}$ for each $n > 0$, and ; we call this new language $\mathcal{L}^{**}_{\mathrm{oab}}(V)$. Let $T^{**}_{e,f}$ be the $\mathcal{L}^{**}_{\mathrm{oab}}(V)$-theory extending $T^*_{e,f}$ by the following defining axioms for the new unary predicates and the new constant symbols (for every $n > 0$)

$$\forall x\big(E_n(x) \leftrightarrow V(x) \wedge \exists y(V(y) \wedge (x = ny))\big), \tag{3.2}$$

$$\Big(\bigwedge_{1 \le i < j \le k(n)} \neg D_n(c_i^{(n)} - c_j^{(n)})\Big) \wedge \forall x\Big(\bigvee_{i=1}^{k(n)} D_n(x - c_i^{(n)})\Big), \tag{3.3}$$

(3.4)

$$\Big(\bigwedge_{i=1}^{l(n)} V(d_i^{(n)}) \wedge \bigwedge_{1 \le i < j \le l(n)} \neg E_n(d_i^{(n)} - d_j^{(n)})\Big) \wedge \forall x\big(V(x) \to \bigvee_{i=1}^{l(n)} E_n(x - d_i^{(n)})\big).$$

Proposition 3.7. *The $\mathcal{L}^{**}_{\mathrm{oab}}(V)$-theory $T^{**}_{e,f}$ has quantifier elimination.*

Proof. We shall use Fact 1. So let (A, G) be a countable model of $T^{**}_{e,f}$ with a proper $\mathcal{L}^{**}_{\text{oab}}(V)$-substructure (A', G') and let (B, H) be an $\aleph_1$-saturated model of $T^{**}_{e,f}$, with an embedding ι of (A', G') into (B, H). We need to extend ι to an embedding of a substructure of (A, G), extending (A', G') properly, into (B, H).

We proceed as in the proof of the fact that $\mathcal{I}$ has the back-and-forth property, however note that here A' is not necessarily a pure subgroup of A, and (A, G) is not $\aleph_1$-saturated. Let $B' := \iota(A')$ and $H' := \iota(G')$.

First assume that there is $a' \in (A' \cap nA) \setminus nA'$ with $n > 0$. Choose n to be minimal such. Let $a \in A \setminus A'$ be such that $a' = na$. Thus $\iota(a') \in B' \cap nB$. Take $b \in B \setminus B'$ such that $\iota(a') = nb$. Note that this b realizes the cut in B' corresponding to the cut of a in A'. Now let $A'' := A' + \mathbb{Z}a$, $G'' := A'' \cap G$, and $B'' := B' + \mathbb{Z}b$. So A'' and B'' are isomorphic as ordered abelian groups, moreover this isomorphism takes G'' into H. Hence ι extends to an $\mathcal{L}^{**}_{\text{oab}}(V)$-embedding of (A'', G'') into (B, H) by taking a to b.

So we may assume that A' is pure in A. We also know that $[p]A' = [p]A$ and $[p]G' = [p]G$ for any p. Let (A^*, G^*) be a κ-saturated elementary extension of (A, G). Thus there is back-and-forth system, $\mathcal{I}$, between (A^*, G^*) and (B, H), and $\iota : (A', G') \to (B', H')$ is in $\mathcal{I}$. Hence for $a \in A$, we can extend ι to a proper extension of (A', G') containing a, moreover this extension can be chosen to be included in (A, G). □

As a result of this proposition, we get:

Corollary 3.8. *The $\mathcal{L}_{\text{oab}}(V)$-theory $T_{\mathbf{0},\mathbf{0}}$ admits quantifier elimination.*

Remark. Instead of adding a unary predicate symbol D_n for $n > 0$, we could have added a unary function symbol $\frac{1}{n}$ for each $n > 0$ to get $\mathcal{L}^{*}_{\text{oab}}(V)$; and we could have extended $T_{e,f}$ to $T^{*}_{e,f}$ by adding the following defining axiom

$$\forall x \forall y \big(\frac{1}{n}x = y \leftrightarrow (x = ny) \vee \forall z (x \neq nz \wedge y = 0)\big), \tag{3.5}$$

for each $n > 0$, instead of (3.1). We would still get the model completeness result 3.6. Also if we had extended $\mathcal{L}^{*}_{\text{oab}}(V)$ to $\mathcal{L}^{**}_{\text{oab}}(V)$ and $T^{*}_{e,f}$ to $T^{**}_{e,f}$ in a similar way, we would get the quantifier elimination result 3.7. Note that when we work with this new $\mathcal{L}^{**}_{\text{oab}}(V)$, the meaning of substructure changes. Indeed if (A', G') is an $\mathcal{L}^{**}_{\text{oab}}(V)$-substructure of (A, G), then A' is a pure subgroup of A, and hence $G' = A' \cap G$ is a pure subgroup of G.

The induced structure on A/G. Let (A, G) be a model of $T^{**}_{e,f}$. Now we consider the structure, $(A/G)_{\text{ind}}$, on A/G induced by the pair (A, G), i.e. the structure with the underlying set A/G, whose definable relations are of the form $\pi(Y)$, where

$$\pi : A \to A/G$$

is the usual projection map, and $Y \subseteq A^n$ is definable in (A, G). For instance, the abelian group structure on A/G is definable in $(A/G)_{\text{ind}}$. We show that the definable relations of $(A/G)_{\text{ind}}$ are the same as the definable relations of the abelian group A/G.

We introduce a 2-sorted language $\mathcal{L}$ with sorts s, t. It has the following nonlogical symbols:

- a function symbol $+$ of arity $(s, s; s)$,
- a function symbol $-$ of arity $(s; s)$,
- a function symbol $\frac{1}{n}$ of arity $(s; s)$ for each $n > 0$,
- a relation symbol $<$ of arity (s, s),
- a relation symbol E_n of arity (s) for each $n > 0$,
- a constant symbol 0 of arity (s),
- $k(n)+l(n)$ many distinct constant symbols of arity (s) for each $n > 0$,
- a function symbol $+$ of arity $(t, t; t)$,
- a function symbol $-$ of arity $(t; t)$,
- a constant symbol 0 of arity (t),
- a function symbol P of arity $(s; t)$,

We consider the symbol $+$ of arity $(s, s; s)$ as distinct from the symbol $+$ of arity $(t, t; t)$; likewise we use $-$ as two distinct symbols, and 0 as well. We denote $\mathcal{L}$-structures as $(A, B; P)$.

An $\mathcal{L}$-homomorphism $f : (A, B; P) \to (C, D; Q)$ consists of two maps

$$f_s : A \to C \text{ and } f_t : B \to D.$$

However in what follows we do not distinguish between these maps, and call both of them f.

Let T be the $\mathcal{L}$-theory whose models are of the form $(A, B; P)$, where A is a regularly dense ordered abelian group, B is an abelian group, and $P : A \to B$ is a surjective abelian group homomorphism such that $\ker P$ is a dense (ordered) subgroup of A, B is infinite, and the system of prime invariants of $(A, \ker P)$ is e, f. The constant symbols in the seventh item are interpreted as the coset representatives for nA in A, and $n \ker P$ in $\ker P$. If $(A, G) \models T_{e,f}$, then $(A, A/G; \pi) \models T$.

We show that T has quantifier elimination using Fact 1. So let $(A, B; P)$, and $(C, D; Q)$ be two models of T such that $(C, D; Q)$ is $|A|^+$-saturated, and let $(A', B'; P')$ be a proper substructure of $(A, B; P)$ with an embedding

$$f : (A', B'; P') \to (C, D; Q).$$

Note that A' is a pure subgroup of A, and hence $\ker P'$ is a pure subgroup of $\ker P$, and thus for any p, $[p]A' = [p]A$ and $[p] \ker P' = [p] \ker P$. Similarly $f(A')$ and $f(\ker P')$ are pure subgroups of C and $\ker Q$ respectively, and for any p, $[p]f(A') = [p]C$ and $[p]f(\ker P') = [p] \ker Q$. Note also that there is an $\mathcal{L}_{\text{oab}}(V)$-embedding

$$F : (A', \ker P') \to (C, \ker Q), \quad a' \mapsto f(a').$$

We wish to extend f to an embedding $(A'', B''; P'') \to (C, D; Q)$, where $(A'', B''; P'')$ is an $\mathcal{L}$-substructure of (A, B, P) such that either $A'' \neq A'$ or $B'' \neq B'$.

First suppose that $B' = B$. We have a couple of subcases.

(i) $\ker P \not\subseteq A'$: Let $a \in \ker P \setminus A'$. Using the back-and-forth system constructed earlier, we can extend F to an $\mathcal{L}_{\mathrm{oab}}(V)$-embedding

$$(A'\langle a\rangle, \ker P'\langle a\rangle) \to (C, \ker Q).$$

Now this gives an $\mathcal{L}$-embedding $(A'\langle a\rangle, B'; P') \to (C, D; Q)$ extending f.

(ii) $\ker P \subseteq A'$: This yields $A'\langle \ker P\rangle = A'$. Let $a \in A \setminus A'$. Put $b := P(a)$, and $d := f(b)$. Note that $b \notin P(A')$, and $d \notin \langle f(P(A'))\rangle_D$. Take a $c \in Q^{-1}(d)$ realizing the cut in $f(A')$ corresponding to the cut of a in A'. Then we can extend f to

$$(A'\langle a\rangle_A, B'; P) \to (C, D; Q),$$

by mapping a to c.

Now we assume that $B' \neq B$. So let $b \in B \setminus B'$. By saturation, we may choose $d \in D \setminus f(B')$ realizing the type over D' in the abelian group D corresponding to the type of b over B' in the abelian group B. Now let $B'' = B' + \mathbb{Z}b$. Then we can extend f to

$$\tilde{f} : (A', B'') \to (C, D),$$

by taking b to d, finishing the proof of the fact that T has quantifier elimination.

Now we consider the original problem, the induced structure on A/G by (A, G), where $(A, G) \models T_{e,f}$. Note that if $X = \pi(Y) \subseteq (A/G)^n$, where $Y \subseteq A^n$ is definable in (A, G), then X is definable in the $\mathcal{L}$-structure $(A, A/G, \pi)$. As $(A, A/G, \pi)$ has quantifier elimination, we get that X is definable in the abelian group A/G.

4. The case that G is regularly discrete

An abelian group G is said to be *regularly discrete* if G has a smallest positive element and $|G/pG| = p$ for every p. It is easy to see that if G has a least positive element, then the following are equivalent:

(1) G is regularly discrete;
(2) for every p and all $g < h$ in G such that (g, h) has at least p elements we have $pG \cap (g, h) \neq \emptyset$;
(3) G is a $\mathbb{Z}$-group.

For these facts, see [7] and [8].

If A is regularly dense and G is an ordered subgroup of A with a smallest positive element 1 such that

(4.1) $\qquad$ for all $a \in A$ there is $g \in G$ such that $g \leq a < g + 1$, then G is regularly discrete and cofinal in A. Note that (4.1) is satisfied if for each $a \in A$, there are $g_1, g_2 \in G$ such that $g_1 \leq a < g_2$, and the interval (g_1, g_2) contains finitely many elements of G.

In the rest of this section A is a regularly dense ordered abelian group, G is a subgroup of A such that G has a smallest positive element 1 *and (4.1) holds for A and G, and $[p]A$ and $[p]G$ are finite for each p.* (These assumptions are clearly satisfied if A is a dense subgroup of $\mathbb{R}^{>0}$ of finite rank and G is a subgroup of A such that G has a smallest positive element, and G is cofinal in A.)

Let B be a regularly dense ordered abelian group and H an ordered subgroup of B with a smallest positive element such that (4.1) holds with B, H in the place of A, G. Also suppose that (A, G) and (B, H) have the same system of prime invariants and are κ-saturated, where κ is an uncountable cardinal. We denote the smallest positive element of G and H by 1.

Let $\mathcal{L}_{\text{oab1}}$ be the language augmenting $\mathcal{L}_{\text{oab}}$ by a constant symbol 1, similarly $\mathcal{L}_{\text{oab1}}(V)$ extends $\mathcal{L}_{\text{oab}}(V)$ by a constant symbol 1.

Let $\mathcal{I}$ be the set of isomorphisms

$$\iota : (A', G') \to (B', H')$$

where (A', G') and (B', H') are $\mathcal{L}_{\text{oab1}}(V)$-substructures of (A, G) and (B, H) of cardinality less than κ such that

(1) A', B' are pure subgroups of A, B respectively,
(2) for any $a' \in A'$, there is $g' \in G'$ such that $g' \leq a' < g' + 1$, and for any $b' \in B'$, there is $h' \in H'$ such that $h' \leq b' < h' + 1$,
(3) for each p,

$$[p]A' \geq [p]A, \quad [p]B' \geq [p]B.$$

Remarks. Let (A', G') be a substructure of (A, G) satisfying (1). Then $G' = A' \cap G$ is a pure subgroup of G. Therefore for each p, $[p]G' \leq [p]G = p$, and since G' has a smallest positive element, namely 1, we get $[p]G' = [p]G = p$. Thus G' is a regularly discrete ordered abelian group. If in addition to (1), A', B' also satisfy (3), then for each p

$$[p]A' = [p]A \text{ and } [p]B' = [p]B.$$

We proceed to prove that $\mathcal{I}$ has the back-and-forth property. So let

$$\iota : (A', G') \to (B', H')$$

be in $\mathcal{I}$. Given $a \in A \setminus A'$, we extend ι to an isomorphism in $\mathcal{I}$ which contains a in its domain.

First let $a \in G$. By Lemma 2.5, we can find $b \in H$ such that for all $a' \in A'$, $g' \in G'$, $m, n > 0$ and $k, l \in \mathbb{Z}$:

$$a' + ka \in mA \Longleftrightarrow \iota(a') + kb \in mB, \tag{4.2}$$

$$g' + la \in nG \iff \iota(g') + lb \in nH. \tag{4.3}$$

Now we modify b such that for each $N \in \mathbb{N}^{>0}$, Nb realizes the cut in B' corresponding to the cut of Na in A'. Employing the arguments in the case that G is regularly dense, it is enough to prove this for a single $N \in \mathbb{N}^{>0}$. Let $a_1', a_2' \in A'$ such that $a_1' < Na < a_2'$. Since $a \notin A'$, using (2), we get

$$a_1' < a_1' + 1 < a_1' + 2 < \cdots < Na < \cdots < a_2' - 2 < a_2' - 1 < a_2'.$$

Thus there are infinitely many elements of G in the interval (a_1', a_2'). So the interval $\big(\iota(a_1') - Nb, \iota(a_2') - Nb\big)$ contains infinitely many elements of H. Therefore as H is regularly discrete, for every $n > 0$,

$$\big(\iota(a_1') - Nb, \iota(a_2') - Nb\big) \cap nH$$

is nonempty. Thus by saturation $\big(\iota(a_1') - Nb, \iota(a_2') - Nb\big)$ contains an element h of H which is divisible by all $n > 0$ in H. Now $Nb + h \in \big(\iota(a_1'), \iota(a_2')\big)$, and for all $a' \in A'$, $g' \in G'$, $m, n > 0$ and $k, l \in \mathbb{Z}$:

$$a' + kNa \in mA \iff \iota(a') + k(Nb + h) \in mB,$$

$$g' + lNa \in nG \iff \iota(g') + l(Nb + h) \in nH.$$

Hence by saturation, b satisfying (4.2) and (4.3) can be chosen in a way that for each $N > 0$, Nb realizes the cut in B' corresponding to the cut of Na in A'. Now we can extend ι to an $\mathcal{L}_{\mathrm{oab1}}(V)$-isomorphism

$$(A'\langle a\rangle_A, G'\langle a\rangle_G) \to (B'\langle b\rangle_B, H'\langle b\rangle_H).$$

As noted in the regularly dense case, (1) and (3) hold with $A'\langle a\rangle_A$,$G'\langle a\rangle_G$, $B'\langle b\rangle_B$, and $H'\langle b\rangle_H$ in the place of A', G', B' and H'. It is also easy to check (2) with $A'\langle a\rangle_A, G'\langle a\rangle_G, B'\langle b\rangle_B$, and $H'\langle b\rangle_H$.

If $a \in A'\langle G\rangle_A$, we extend ι to the desired isomorphism exactly as in the case that G is regularly dense.

Now let $a \in A \setminus A'\langle G\rangle_A$. Let $g \in G$ be such that $g \le a < g + 1$. Using the first case, choose $h \in H$ such that ι extends to an isomorphism

$$\tilde{\iota} : \big(A'\langle g\rangle_A, G'\langle g\rangle_G\big) \to \big(B'\langle h\rangle_B, H'\langle h\rangle_H\big).$$

Now let $b \in B \setminus B'\langle H\rangle_B$ such that for each $N \in \mathbb{N}^{>0}$, Nb realizes the cut in B' corresponding to the cut of Na in A'. So we can extend $\tilde{\iota}$ to an isomorphism

$$\big(A'\langle a, g\rangle_A, G'\langle g\rangle_G\big) \to \big(B'\langle b, h\rangle_B, H'\langle h\rangle_H\big).$$

It is again easy to check that (1) and (3) are satisfied for $A'\langle a, g\rangle_A, G'\langle g\rangle_G$, $B'\langle b, h\rangle_B, H'\langle h\rangle_H$ in the place of A', G', B', H'. So it remains to show (2) with $A'\langle a, g\rangle_A, G'\langle g\rangle_G, B'\langle b, h\rangle_B$ and $H'\langle h\rangle_H$ in the place of A', G', B' and H'. Let $\frac{a'+ka+lg}{m} \in A'\langle a, g\rangle$. Note that we may assume that $k \ge 0$. Let $g' \in G'$ be such that $g' \le a' < g' + 1$. Hence

$$g' + (k + l)g \le a' + ka + lg < g' + (k + l)g + (k + 1).$$

Then we choose i, j from $\{0, 1, \ldots, m\}$ such that $g' + (k+l)g - i \in mG$, and $g' + (k+l)g + (k+1) + j \in mG$. Hence

$$\frac{g' + (k+l)g - i}{m} \leq \frac{a' + ka + lg}{m} < \frac{g' + (k+l)g + (k+1) + j}{m}.$$

Thus we get (2), as there are finitely many elements of $G'\langle g\rangle$ between $\frac{g'+(k+l)g-i}{m}$ and $\frac{g'+(k+l)g+(k+1)+j}{m}$. This finishes the proof of the fact that $\mathcal{I}$ has the back-and-forth property.

Now we prove a series of results analogous to the results in the previous section.

Corollary 4.1. *Let (A', G') be an $\mathcal{L}_{\text{oab1}}(V)$-substructure of (A, G), where A' is pure in A, and regularly dense, and G' is a regularly discrete subgroup of A' such that for each $a' \in A'$, there is $g' \in G'$ with $g' \leq a' < g' + 1$. Suppose that (A, G) and (A', G') have the same system of prime invariants. Then $(A', G') \preceq (A, G)$.*

Proof. Let κ be an uncountable cardinal larger than $|A'|$, and let (A^*, G^*) be a κ-saturated elementary extension of (A', G'). We may assume that (A, G) is also κ-saturated. Then we have a back-and-forth system $\mathcal{I}$ between (A, G) and (A^*, G^*) containing the identity map on (A', G'), which yields the desired result. $\square$

For a family $e = \big(e(p)\big)$ of natural numbers indexed by the primes, let T_e be the $\mathcal{L}_{\text{oab1}}(V)$-theory whose models are of the form (A, G) where A is a regularly dense ordered abelian group, G is a subgroup of A with the smallest positive element 1 such that for any $a \in A$, there is $g \in G$ with $g \leq a < g + 1$, and A/G is infinite, and the system of prime invariants of (A, G) is $e, \mathbf{1}$. By an earlier remark if (A, G) is a model of T_e, then G is automatically regularly discrete.

To construct a model of T_e, first assume that e is not identically 0. Let $A = \bigoplus_p \mathbb{Z}_{(p)}^{e(p)}$, and $G = \mathbb{Z}$. Consider G as a subgroup of A by embedding G in the first copy of $\mathbb{Z}_{(p)}$, where p is the smallest prime such that $e(p) \neq 0$. Embed A into the additive group of $\mathbb{R}$, and equip it with the order induced from $\mathbb{R}$. Then (A, G) becomes a model of T_e.

If $e(p) = 0$ for each p, then take $A = \mathbb{Q}$ with the order induced from $\mathbb{R}$, and $G = \mathbb{Z}$. Then (A, G) is a model of T_e.

Thus T_e is consistent for each $e = \big(e(p)\big)$.

Let $x = (x_1, \ldots, x_m)$ be a tuple of variables. Recall that in the previous subsection a special formula in x was defined to be an $\mathcal{L}_{\text{oab}}(V)$-formula $\exists y\big(V(y) \wedge \theta_V(y) \wedge \psi(x, y)\big)$, where $y = (y_1, \ldots, y_n)$, and $\theta(y)$ and $\psi(x, y)$ are $\mathcal{L}_{\text{oab}}$-formulas. In this subsection a *special formula in x* is an $\mathcal{L}_{\text{oab1}}(V)$ formula $\exists y\big(V(y) \wedge \theta_V(y) \wedge \psi(x, y)\big)$, where $\theta(y)$ is an $\mathcal{L}_{\text{oab1}}$-formula, and $\psi(x, y)$ is an $\mathcal{L}_{\text{oab}}$-formula.

Lemma 4.2. *Each $\mathcal{L}_{\text{oab}}(V)$-formula is $T_{\mathbf{0}}$-equivalent to a boolean combination of special formulas.*

Proof. Let (A,G) and (B,H) be two $\aleph_1$-saturated models of $T_{\mathbf{0}}$; thus taking $\kappa=\aleph_1$ there is a back-and-forth system $\mathcal{I}$ between (A,G) and (B,H). Let $a\in A^m$ and $b\in B^m$ satisfy the same special formulas. It suffices to prove that a and b satisfy the same types in (A,G) and (B,H) respectively. To show this, it is enough to find an element $\iota:(A',G')\to(B',H')$ of $\mathcal{I}$ with $a\in(A')^m$, $b\in(B')^m$ and $\iota(a)=b$.

Let $\text{rk}(G\langle a\rangle_A/G)=r$. We may assume that $\text{rk}(G\langle a_1,\dots,a_r\rangle_A/G)=r$, and $a_i\in G\langle a_1,\dots,a_r\rangle_A$ for $r<i\le m$. Then since a and b satisfy the same special formulas, we have

$$\text{rk}(H\langle b\rangle_B/H)=r \text{ and } b_i\in H\langle b_1,\dots,b_r\rangle_B \text{ for } r<i\le m.$$

Take $g'_1,\dots,g'_r\in G$ such that $g'_i\le a_i<g'_i+1$ for all $i=1,\dots,r$. Now let G' be a countable pure subgroup of G containing $g'_1,\dots,g'_r$ and 1 such that $\text{rk}(G'\langle a\rangle_A/G')=r$. Note that $G'\langle a\rangle_A$ is divisible, being a pure subgroup of the divisible group A. Thus $[p]G'\langle a\rangle_A=[p]A=1$ for every p. Hence $(G'\langle a\rangle_A,G')$ is a countable substructure of (A,G) satisfying (1), (2) and (3)(with $G'\langle a\rangle_A$ in the place of A').

Enumerate G' as $g=(g_0,g_1,\dots)$, and let $y=(y_0,y_1,\dots)$ be a countable tuple of distinct variables. If $\theta_1(y),\dots,\theta_k(y)$ are $\mathcal{L}_{\text{oab1}}$-formulas and $\phi_1(x,y),\dots,\ \phi_k(x,y)$ are $\mathcal{L}_{\text{oab}}$-formulas such that $G\models\theta_i(g)$ and $A\models\phi_i(a,g)$ for $i=1,\dots,k$, then

$$(A,G)\models\exists y\big(V(y)\wedge\theta_V(y)\wedge\phi(a,y)\big),$$

where $\theta(y):=\bigwedge_{i=1}^k\theta_i(y)$, and $\phi(x,y):=\bigwedge_{i=1}^k\phi_i(x,y)$. Therefore

$$(B,H)\models\exists y\big(V(y)\wedge\theta_V(y)\wedge\phi(b,y)\big).$$

So we have a partial y-type over b in (B,H), consisting of formulas $\theta_V(y)$ and $\phi(b,y)$ where $G\models\theta(g)$ and $A\models\phi(a,g)$. Thus by saturation there is a countable tuple $h=(h_0,h_1,\dots)$ realizing this type. Now let H' be $\{h_0,h_1,\dots\}$. Then H' is a countable pure subgroup of H containing 1 which is $\mathcal{L}_{\text{oab1}}$-isomorphic to G' such that $\text{rk}(H'\langle b\rangle_B/H')=r$, $[p]H'=[p]H$ for every p, and $H'\langle b\rangle_B$ is divisible. Hence $(H'\langle b\rangle_B,H')$ is a countable substructure of (B,H) satisfying (1), (2), and (3)(with $H'\langle b\rangle_B$ in the place of B'). Moreover there is an ordered group isomorphism $G'\langle a\rangle_A\to H'\langle b\rangle_B$ sending g_i to h_i for $i=0,1\dots$, and a to b. Now the $\mathcal{L}_{\text{oab1}}(V)$-isomorphism

$$\iota:(G'\langle a\rangle_A,G')\to(H'\langle b\rangle_B,H')$$

is in $\mathcal{I}$. □

We prove the next corollary and the proposition after it by using the following well-known fact.

Fact. The theory of $\mathbb{Z}$-groups admits quantifier elimination in the language $\mathcal{L}_{\text{oab1}}$ augmented by a new unary predicate E_n for each $n>0$, interpreted in each $\mathbb{Z}$-group G as the subgroup nG. In particular the theory of $\mathbb{Z}$-groups is complete.

The following consequence of Proposition 4.2 is already proven in [5].

Corollary 4.3. *The $\mathcal{L}_{\text{oab}}(V)$-theory $T_{\mathbf{0}}$ is complete.*

Proof. Let σ be an $\mathcal{L}_{\text{oab}}(V)$-sentence. Then by Lemma 4.2, we may assume that σ is of the form

$$\exists y\big(V(y) \wedge \theta_V(y) \wedge \psi(y)\big),$$

where $\theta(y)$ is an $\mathcal{L}_{\text{oab1}}$-formula and $\psi(y)$ is an $\mathcal{L}_{\text{oab}}$-formula. Moreover we may choose $\psi(y)$ to be quantifier-free since the theory of divisible ordered abelian groups admit quantifier elimination. Thus σ holds true in a model (A, G) of $T_{\mathbf{0}}$ if and only if $\exists y(\theta(y) \wedge \psi(y))$ holds true in G. As the theory of $\mathbb{Z}$-groups is complete, the completeness of $T_{\mathbf{0}}$ follows. □

Proposition 4.4. *Let (A, G) be a model of $T_{\mathbf{0}}$. Then every subset of A^m definable in (A, G) is a boolean combination of subsets of A^m defined in $(A, G)$by formulas $\exists y\big(V(y) \wedge \psi(x, y)\big)$, where $\psi(x, y)$ is a quantifier-free $\mathcal{L}_{\text{oab}}(A)$-formula(here $\mathcal{L}_{\text{oab}}(A)$ is the language of ordered abelian groups augmented by names for the elements of A).*

Proof. By Lemma 4.2, it is enough to prove that the subsets of A^m defined by special formulas are of the required form. Let $\theta(y)$ be an $\mathcal{L}_{\text{oab1}}$-formula and $\phi(x, y)$ an $\mathcal{L}_{\text{oab}}$-formula. Our task is to find a quantifier-free $\mathcal{L}_{\text{oab}}(A)$-formula $\psi(x, z)$ with z a tuple of variables such that $\big(\exists y\big(V(y) \wedge \theta_V(y) \wedge \phi(x, y)\big)$ is equivalent in (A, G) to $\exists z\big(V(z) \wedge \psi(x, z)\big)\big)$.

By the fact above, the set $\{g \in G^n : G \models \theta(g)\}$ is a boolean combination of subsets of G^n of the form

$$\{g \in G^n : k_1 g_1 + \cdots + k_n g_n = N\},$$
$$\{g \in G^n : k_1 g_1 + \cdots + k_n g_n < N\},$$
$$\{g \in G^n : k_1 g_1 + \cdots + k_n g_n \in dG\},$$

where $k_1, \ldots, k_n \in \mathbb{Z}$, $N \in \mathbb{N}$ and d is a positive integer.

Since G is regularly discrete $|G/dG|$ is an integer, and hence the complement of the set

$$\{g \in G^n : k_1 g_1 + \cdots + k_n g_n \in dG\}$$

is a finite union of sets of the form

$$\{g \in G^n : k_1 g_1 + \cdots + k_n g_n \in g_0 + dG\},$$

where $g_0 \in G$.

Thus we get the desired $\psi(x, z)$, which can be chosen to be quantifier-free as the theory of divisible ordered abelian groups admit quantifier elimination. □

Now we add a unary predicate D_n for each $n > 0$ to $\mathcal{L}_{\mathrm{oab1}}(V)$ to get the language $\mathcal{L}^*_{\mathrm{oab1}}(V)$. Extend T_e to an $\mathcal{L}^*_{\mathrm{oab1}}(V)$-theory T^*_e by adding the defining axiom (3.1) for each $n > 0$.

Corollary 4.5. *The $\mathcal{L}^*_{\mathrm{oab1}}(V)$-theory T^*_e is model complete.*

Proof. Let $(A', G') \subseteq (A, G)$ be two models of T^*_e. Then by 3.4, we get that $A' \preceq A$. In particular A' is pure in A. Now by Corollary 4.1, $(A', G') \preceq (A, G)$. □

Next for each p, add $p^{e(p)}$ many distinct constant symbols, and a unary function λ to $\mathcal{L}^*_{\mathrm{oab1}}(V)$ to get the language $\mathcal{L}^{**}_{\mathrm{oab1}}(V)$, and extend T^*_e to an $\mathcal{L}^{**}_{\mathrm{oab1}}(V)$-theory by adding by adding defining axioms (3.3) for the new constant symbols, and the following axiom for λ

$$\lambda(x) = y \leftrightarrow \big(V(y) \wedge (y \leq x < y + 1)\big).$$

Note that if (A, G) is a model of T^{**}_e, then λ is a well-defined function whose image is G.

Proposition 4.6. *The $\mathcal{L}^{**}_{\mathrm{oab1}}(V)$-theory T^{**}_e admits quantifier elimination.*

Proof. We apply the fact mentioned in the introduction. So let (A, G), and (B, H) be two models of T^{**}_e, such that (A, G) is countable and (B, H) is $\aleph_1$-saturated, also let (A', G') be a $\mathcal{L}^{**}_{\mathrm{oab1}}(V)$-substructure of (A, G) with an embedding $\iota : (A', G') \to (B, H)$. Let $B'\iota(A')$ and $H' = \iota(G')$. We need to extend ι properly to an embedding of an $\mathcal{L}^{**}_{\mathrm{oab1}}(V)$-substructure of (A, G). If A' is pure in A, then we can extend ι properly using the back-and-forth system constructed earlier in this section. If not, then take $a' \in A'$ such that $a' \in nA \setminus A'$. Then $\iota(a') \in nB \setminus B'$. Note that b realizes the cut in B' corresponding to the cut of a in A'. Thus there is an ordered group isomorphism $A'\langle a\rangle_A \to B'\langle b\rangle_B$ which is easily an $\mathcal{L}^{**}_{\mathrm{oab1}}(V)$-isomorphism. □

5. Real closed fields with two multiplicative subgroups

In this section we use the notation of [3]. In particular $\mathcal{L}_{\mathrm{o}}$ is the language of ordered rings with the sublanguage $\mathcal{L}_{\mathrm{om}}$ of ordered monoids. We use $\mathcal{L}_{\mathrm{o}}(U, V)$ for the enrichment of $\mathcal{L}_{\mathrm{o}}$ by two distinct unary predicate symbols U, V.

Fix a real closed field R, and two subgroups Δ, Γ of $R^{>0}$ with the Mann property such that $\Gamma \subseteq \Delta$, and $[p]\Delta$, and $[p]\Gamma$ are finite for each p. Thus for any $n > 0$, $[n]\Delta$ and $[n]\Gamma$ are finite, say $k_n := [n]\Delta$ and $l_n := [n]\Gamma$. For every $n > 0$, fix sets $\{\delta_{n1}, \ldots, \delta_{nk_n}\}$, and $\{\gamma_{n1}, \ldots, \gamma_{nl_n}\}$ of coset representatives for $\Delta^{[n]}$ in Δ, and $\Gamma^{[n]}$ in Γ respectively.

Now let $\mathcal{L}_{\mathrm{o}}(\Delta, U, V)$ be the language obtained by augmenting $\mathcal{L}_{\mathrm{o}}(U, V)$ by a constant symbol δ' for each element δ of Δ. We define the *ordering* and *Mann axioms* of Δ and Γ as in [3].

We first prove a couple of lemmas.

Lemma 5.1. *Let K be a field with subgroups A, G of $K^{>0}$ with $G \subseteq A$, and let K' be a subfield of K with subgroups A', G' of $(K')^{>0}$ such that $A' \subseteq A$ and $G' = A' \cap G$. Suppose that for all $q_1, \ldots, q_n \in \mathbb{Q}$ the equation $q_1x_1 + \cdots + q_nx_n = 1$ has the same nondegenerate solutions in A' as in A, and that K' and $\mathbb{Q}(A)$ are free over $\mathbb{Q}(A')$. Then K' and $\mathbb{Q}(G)$ are free over $\mathbb{Q}(G')$.*

Proof. Let $g_1, \ldots, g_m \in G$ be algebraically dependent over K'. Since K' and $\mathbb{Q}(A)$ are free over $\mathbb{Q}(A')$, $g_1, \ldots, g_m$ are algebraically dependent over $\mathbb{Q}(A')$. So there are nonzero integers $k_1, \ldots, k_n$, $a_1, \ldots, a_n \in A'$, and multi-indices $i_1, \ldots, i_n$ of length m such that $k_1a_1g^{i_1} + \cdots + k_na_ng^{i_n} = 0$, where $g = (g_1, \ldots, g_m)$. We also assume that n is minimal such. As a result of this

$$(\frac{a_1g^{i_1}}{a_ng^{i_n}}, \ldots, \frac{a_{n-1}g^{i_n-1}}{a_ng^{i_n}})$$

is a nondegenerate solution of $(-\frac{k_1}{k_n})x_1 + \cdots + (-\frac{k_{n-1}}{k_n})x_{n-1} = 1$ in A. Thus we get $\frac{a_1g^{i_1}}{a_ng^{i_n}} \in A'$. Therefore $g^{i_1-i_n}$ is in A', hence in G'. So $g_1, \ldots, g_k$ are algebraically dependent over $\mathbb{Q}(G')$, which proves that K' and $\mathbb{Q}(G)$ are free over $\mathbb{Q}(G')$. □

By Lemmas 5.12, and 5.13 of [3], we get the following:

Lemma 5.2. *Let K, A, G, K', A', G' be as in the previous lemma. Suppose also that A' is pure in A. Then (K', A', G') is a substructure of (K, A, G).*

Remark. As noted in [3], in the setting of the lemma above it follows that $\mathbb{Q}(A)$ is a regular extension of $\mathbb{Q}(A')$, and that $\mathbb{Q}(G)$ is a regular extension of $\mathbb{Q}(G')$. Hence K' and $\mathbb{Q}(A)$ are linearly disjoint over $\mathbb{Q}(A')$, and K' and $\mathbb{Q}(G)$ are linearly disjoint over $\mathbb{Q}(G')$.

5.1. **The case that G is dense.** Let $\mathrm{RCF}_1(\Delta, \Gamma)$ be the $\mathcal{L}_{\mathrm{o}}(\Delta, U, V)$-theory whose models are of the form $\big(K, A, G, (\delta')_{\delta\in\Delta}\big)$ such that:

(1) K is a real closed field, G and A are dense subgroups of $K^{>0}$ such that $G \subseteq A$, and $|A/G|$ is infinite,
(2) $\delta \mapsto \delta' : \Delta \to A$ and $\gamma \mapsto \gamma' : \Gamma \to G$ are group homomorphisms, and if $\delta' \in G$, then $\delta \in \Gamma$,
(3) For any $n > 0$, and $a \in A$, one of $a^{-1}\delta'_{n1}, \ldots, a^{-1}\delta'_{nk_n}$ is in $A^{[n]}$,
(4) For any $n > 0$, and $g \in G$, one of $g^{-1}\gamma'_{n1}, \ldots, g^{-1}\gamma'_{nl_n}$ is in $G^{[n]}$,
(5) $\big(K, (\delta')_{\delta\in\Delta}\big)$ satisfies the ordering axioms for Δ,
(6) $\big(K, A, (\delta')_{\delta\in\Delta}\big)$ satisfies the Mann axioms for Δ,
(7) $\big(K, G, (\gamma')_{\gamma\in\Gamma}\big)$ satisfies the Mann axioms for Γ.

So if $\big(K, A, G, (\delta')_{\delta\in\Delta}\big)$ is a model of $\mathrm{RCF}_1(\Delta, \Gamma)$, then Δ and Γ are embedded into A and G respectively as ordered groups. From now on, we identify the images of Δ and Γ in K by themselves, and write a model of $\mathrm{RCF}_1(\Delta, \Gamma)$ as $\big(K, A, G, (\delta)_{\delta\in\Delta}\big)$ or $\big(K, A, G, (\delta)\big)$.

A dense multiplicative subgroup of $K^{>0}$ is regularly dense whenever K is a real closed field. Combining this fact with (1), (3) and (4), the results of the first subsection of Section 3 can be applied to (A, G) whenever (K, A, G) is a model of $\mathrm{RCF}_1(\Delta, \Gamma)$.

If Δ and Γ are dense in R, and $|\Delta/\Gamma|$ is infinite, then the structure $\big(R, \Delta, \Gamma, (\delta)_{\delta\in\Delta}\big)$ is a model of $\mathrm{RCF}_1(\Delta, \Gamma)$. Also note that $\mathrm{RCF}_1(\Delta, \Gamma)$ is complete if and only if both Δ and Γ are divisible.

We classify the models of $\mathrm{RCF}_1(\Delta, \Gamma)$ up to elementary equivalence.

Theorem 5.3. *Let $\big(K, A, G, (\delta)_{\delta\in\Delta}\big)$ and $\big(L, B, H, (\delta)_{\delta\in\Delta}\big)$ be two models of $\mathrm{RCF}_1(\Delta, \Gamma)$. Then they are elementarily equivalent if and only (A, G) and (B, H) have the same system of prime invariants, and for all $\delta \in \Delta$, $\gamma \in \Gamma$, and $m, n > 0$:*

$$\delta \textit{ is an } m^{\text{th}} \textit{ power in } A \Longleftrightarrow \delta \textit{ is an } m^{\text{th}} \textit{ power in } B,$$

$$\gamma \textit{ is an } n^{\text{th}} \textit{ power in } G \Longleftrightarrow \gamma \textit{ is an } n^{\text{th}} \textit{ power in } H.$$

Proof. The forward direction is clear, so we only prove the converse direction. For this purpose, assume that $[p]A = [p]B$, $[p]G = [p]H$ for every p, and for all $\delta \in \Delta$, $\gamma \in \Gamma$, and $m, n > 0$:

$$\delta \text{ is an } m^{\text{th}} \text{ power in } A \Longleftrightarrow \delta \text{ is an } m^{\text{th}} \text{ power in } B,$$

$$\gamma \text{ is an } n^{\text{th}} \text{ power in } G \Longleftrightarrow \gamma \text{ is an } n^{\text{th}} \text{ power in } H.$$

We show that $\big(K, A, G, (\delta)\big)$ and $\big(L, B, H, (\delta)\big)$ are elementarily equivalent by constructing a nonempty back-and-forth system between them. We may assume that $\big(K, A, G, (\delta)\big)$ and $\big(L, B, H, (\delta)\big)$ are κ-saturated for some infinite cardinal $\kappa > |\Delta|$.

Define $\mathrm{Sub}(K, A, G)$ to be the set of all $\mathcal{L}_{\mathrm{o}}(U, V)$-structures (K', A', G') where K' is a real closed subfield of K of cardinality less than κ, A', G' are subgroups of A, G containing Δ, Γ respectively such that A' is pure in A, $G' = A' \cap G$, and K' and $\mathbb{Q}(A)$ are free over $\mathbb{Q}(A')$.

Suppose that $(K', A', G') \in \mathrm{Sub}(K, A, G)$. Then by Lemmas 5.1 and 5.2, we know that K' and $\mathbb{Q}(G)$ are free over $\mathbb{Q}(G')$, and that (K', A', G') is a substructure of (K, A, G). Also it follows from Axioms (3) and (4) that for each p, $[p]A' \geq [p]A$ and $[p]G' \geq [p]G$, and hence $[p]A' = [p]A$ and $[p]G' = [p]G$ by purity.

Define $\mathrm{Sub}(L, B, H)$ in a similar way, and let $\mathcal{I}$ be the set of all isomorphisms $\iota : (K', A', G') \to (L', B', H')$ between elements of $\mathrm{Sub}(K, A, G)$ and $\mathrm{Sub}(L, B, H)$ fixing elements of Δ pointwise. We show that $\mathcal{I}$ is a nonempty back-and-forth system.

We first show that $\mathcal{I} \neq \emptyset$: Let

$$\begin{aligned} A' &:= \{a \in A : a^n \in \Delta \text{ for some } n > 0\}, \quad K' := \mathbb{Q}(\Delta)^{\mathrm{rc}} \subseteq K \\ B' &:= \{b \in B : b^n \in \Delta \text{ for some } n > 0\}, \quad L' := \mathbb{Q}(\Delta)^{\mathrm{rc}} \subseteq L. \\ G' &:= \{g \in G : g^n \in \Gamma \text{ for some } n > 0\}, \\ H' &:= \{h \in H : h^n \in \Gamma \text{ for some } n > 0\}. \end{aligned}$$

Then by axiom (2), $A' \cap G = G'$ and $B' \cap H = H'$, and thus $(K', A', G') \in \mathrm{Sub}(K, A, G)$ and $(L', B', H') \in \mathrm{Sub}(L, B, H)$, and the the ordered field isomorphism $K' \cong L'$ fixing Δ pointwise is in $\mathcal{I}$.

Now we proceed to prove that $\mathcal{I}$ is a back-and-forth system. Let

$$\iota : (K', A', G') \to (L', B', H')$$

be in $\mathcal{I}$ and $\alpha \in K \setminus K'$. We need to find an element of $\mathcal{I}$ with α in its domain. We distinguish some cases.

(I) Let $\alpha \in G$. By the remarks made in Section 3, we can find $\beta \in H$ realizing the cut in B' corresponding to the cut of α in A' such that for all $a' \in A'$, $g' \in G'$, $m, n > 0$ and $k, l \in \mathbb{Z}$:

$$a'\alpha^k \text{ is an } m^{\text{th}} \text{ power in } A \iff \iota(a')\beta^k \text{ is an } m^{\text{th}} \text{ power in } B.$$

$$g'\alpha^l \text{ is an } n^{\text{th}} \text{ power in } G \iff \iota(g')\beta^l \text{ is an } n^{\text{th}} \text{ power in } H.$$

Since B and H are dense in $L^{>0}$ we can choose β to realize the cut in L' corresponding to the cut of α in K'. Now we can extend ι to

$$\big(K'(\alpha)^{\mathrm{rc}}, A'\langle\alpha\rangle_A, G'\langle\alpha\rangle_G\big) \to \big(L'(\beta)^{\mathrm{rc}}, B'\langle\beta\rangle_B, H'\langle\beta\rangle_H\big), \quad \alpha \mapsto \beta.$$

By Lemma 5.12 of [3], this map is in $\mathcal{I}$.

(II) Let $\alpha \in A'\langle G\rangle$. Say $a = (a'g)^{1/m}$ for some $a' \in A', g \in G$ and $m > 0$ such that $a'g \in A^{[m]}$. Then by the previous step there is $h \in H$, and an isomorphism $\big(K'(g)^{\mathrm{rc}}, A'\langle g\rangle_A, G'\langle g\rangle_G\big) \to \big(L'(h)^{\mathrm{rc}}, B'\langle h\rangle_B, H'\langle h\rangle_H\big)$ in $\mathcal{I}$. This finishes this case since $\alpha \in K'(g)^{\mathrm{rc}}$.

(III) Let $\alpha \in A \setminus A'\langle G\rangle$. Then there is $\beta \in B \setminus B'\langle H\rangle$ realizing the cut in B' corresponding to the cut of α in A'

Since A is dense in $K^{>0}$, we can choose β to realize the cut in L' corresponding to the cut of α in K'. So we have an isomorphism

$$\big(K'(\alpha)^{\mathrm{rc}}, A'\langle\alpha\rangle_A, G'\big) \to \big(L'(\beta)^{\mathrm{rc}}, B'\langle\beta\rangle_B, H'\big),$$

which happens to be in $\mathcal{I}$ again by Lemma 5.12 of [3].

(IV) Let $\alpha \in K'(A)^{\mathrm{rc}}$. Then $\alpha \in K'(a_1, \ldots, a_k)^{\mathrm{rc}}$ for some $a_1, \ldots, a_k \in A$. Therefore we can apply some of the previous steps k times in succession to get the desired isomorphism.

(V) Let $\alpha \notin K'(A)^{\mathrm{rc}}$. By saturation and using that B is small (as in [3]), $L \setminus L'(B)^{\mathrm{rc}}$ is dense in L. Thus we may choose $\beta \notin L'(B)^{\mathrm{rc}}$ realizing the cut in L' corresponding to the cut of α in K'. In this case we have an isomorphism $(K'(\alpha)^{\mathrm{rc}}, A', G') \to (L'(\beta)^{\mathrm{rc}}, B', H')$ which is again in $\mathcal{I}$. This finishes the last case and the proof of the theorem. □

Remark. Let Δ and Γ be divisible. If (K, A, G) is a model of $\mathrm{RCF}_1(\Delta, \Gamma)$, then A, G are necessarily divisible. Thus it follows that $\mathrm{RCF}_1(\Delta, \Gamma)$ is complete.

This proof of characterization of models of $\mathrm{RCF}_1(\Delta, \Gamma)$ has the following consequence:

Corollary 5.4. *Let $(K', A', G') \subseteq (K, A, G)$ be $\mathcal{L}_{\mathrm{o}}(U, V)$-structures such that A', G' are subgroups of $(K')^{>0}$ satisfying the Mann property such that $[p]A'$ and $[p]G'$ are finite for each p. Suppose that $\big(K, A, G, (a')_{a' \in A'}\big)$, and $\big(K', A', G', (a')_{a' \in A'}\big)$ are models of $\mathrm{RCF}_1(A', G')$, and that A' and G' are pure subgroups of A and G respectively . Then $(K', A', G') \preceq (K, A, G)$.*

Proof. Let (K^*, A^*, G^*) be a κ-saturated elementary extension of (K', A', G') where κ is an uncountable cardinal greater than $|K'|$. We may also assume that (K, A, G) is κ-saturated. By the proof of 5.3, there is a back-and-forth system $\mathcal{I}$ between substructures of (K^*, A^*, G^*), and (K, A, G) containing the identity map on (K', A', G'). Thus

$$(K^*, A^*, G^*) \equiv_{K'} (K, A, G),$$

and hence $(K', A', G') \preceq (K, A, G)$. □

5.2. **The case that G is regularly discrete.** For the rest of this section we assume that Γ has a smallest element, γ_0, greater than 1 and that Γ is cofinal in Δ.

Let $\mathrm{RCF}_2(\Delta, \Gamma)$ be the $\mathcal{L}_{\mathrm{o}}(\Delta, U, V)$-theory whose models are of the form $\big(K, A, G, (\delta')_{\delta \in \Delta}\big)$ such that:

(1) K is a real closed field, G and A are subgroups of $K^{>0}$ with $G \subseteq A$, A is dense in $K^{>0}$, and G is regularly discrete with smallest element, γ_0' greater than 1,
(2) $\delta \mapsto \delta' : \Delta \to A$ and $\gamma \mapsto \gamma' : \Gamma \to G$ are group homomorphisms, and if $\delta' \in G$, then $\delta \in \Gamma$,
(3) G is cofinal in A,
(4) For any $n > 0$, and $a \in A$, one of $a^{-1}\delta'_{n1}, \dots, a^{-1}\delta'_{nk_n}$ is in $A^{[n]}$,
(5) $\big(K, (\delta')_{\delta \in \Delta}\big)$ satisfies the ordering axioms for Δ,
(6) $\big(K, A, (\delta')_{\delta \in \Delta}\big)$ satisfies the Mann axioms for Δ,
(7) $\big(K, G, (\gamma')_{\gamma \in \Gamma}\big)$ satisfies the Mann axioms for Γ.

As for $\mathrm{RCF}_1(\Delta, \Gamma)$, models of $\mathrm{RCF}_2(\Delta, \Gamma)$ contain copies of Δ and Γ, and we write models of $\mathrm{RCF}_2(\Delta, \Gamma)$ as $\big(K, A, G, (\delta)_{\delta \in \Delta}\big)$ or $\big(K, A, G, (\delta)\big)$. If $\big(K, A, G, (\delta)\big)$ is a model of $\mathrm{RCF}_2(\Delta, \Gamma)$, then A is regularly dense, and G is regularly discrete. Thus Section 4 applies to the pair (A, G).

If Δ is dense in R, and Γ is regularly discrete, then $\big(R, \Delta, \Gamma, (\delta)\big)$ is a model of $\mathrm{RCF}_2(\Delta, \Gamma)$.

We classify the models of $\mathrm{RCF}_2(\Delta, \Gamma)$ up to elementary equivalence as follows.

Theorem 5.5. *Let $\big(K, A, G, (\delta)_{\delta\in\Delta}\big)$ and $\big(L, B, H, (\delta)_{\delta\in\Delta}\big)$ be two models of $\mathrm{RCF}_2(\Delta, \Gamma)$. Then they are elementarily equivalent if and only if for every p, $[p]A = [p]B$, and for all $\delta \in \Delta$, $\gamma \in \Gamma$, and $m, n > 0$:*

$$\delta \textit{ is an } m^{\text{th}} \textit{ power in } A \Longleftrightarrow \delta \textit{ is an } m^{\text{th}} \textit{ power in } B,$$

$$\gamma \textit{ is an } n^{\text{th}} \textit{ power in } G \Longleftrightarrow \gamma \textit{ is an } n^{\text{th}} \textit{ power in } H.$$

Proof. The conclusion is easily satisfied, if $\big(K, A, G, (\delta)\big)$ and $\big(L, B, H, (\delta)\big)$ are elementarily equivalent. So let $\big(K, A, G, (\delta)\big)$ and $\big(L, B, H, (\delta)\big)$ be such that for every p, $[p]A = [p]B$, and for all $\delta \in \Delta$, $\gamma \in \Gamma$, and $m, n > 0$:

$$\delta \text{ is an } m^{\text{th}} \text{ power in } A \Longleftrightarrow \delta \text{ is an } m^{\text{th}} \text{ power in } B,$$

$$\gamma \text{ is an } n^{\text{th}} \text{ power in } G \Longleftrightarrow \gamma \text{ is an } n^{\text{th}} \text{ power in } H.$$

We also assume that they are κ-saturated for some infinite cardinal κ greater than $|\Delta|$. In order to show that they are elementarily equivalent, we proceed in analogy with the proof of Theorem 5.3. So define $\mathrm{Sub}(K, A, G)$ to be the set of all $\mathcal{L}_{\mathrm{o}}(U, V)$-structures (K', A', G') such that K' is a real closed subfield of K of cardinality less than κ, A' is a pure subgroup of A containing Δ, $G' = A' \cap G$, G' is cofinal in A' and K' and $\mathbb{Q}(A)$ are free over $\mathbb{Q}(A')$. Note that by Lemmas 5.1 and 5.2, these assumptions yield that K' and $\mathbb{Q}(G)$ are free over $\mathbb{Q}(G')$, and that $(K', A', G') \subseteq (K, A, G)$. Also by Axiom (4), we get that $[p]A' \geq [p]A$, and hence $[p]A' = [p]A$ since A' is pure in A.

Define $\mathrm{Sub}(L, B, H)$ in a similar way, and let $\mathcal{I}$ be the set of all isomorphisms $\iota : (K', A', G') \to (L', B', H')$ between elements of $\mathrm{Sub}(K, A, G)$ and $\mathrm{Sub}(L, B, H)$ fixing elements of Δ pointwise. We show that $\mathcal{I}$ is a nonempty back-and-forth system.

Let

$$\begin{aligned}
A' &:= \{a \in A : a^n \in \Delta \text{ for some } n > 0\}, \quad K' := \mathbb{Q}(\Delta)^{\mathrm{rc}} \subseteq K \\
B' &:= \{b \in B : b^n \in \Delta \text{ for some } n > 0\}, \quad L' := \mathbb{Q}(\Delta)^{\mathrm{rc}} \subseteq L. \\
G' &:= \{g \in G : g^n \in \Gamma \text{ for some } n > 0\}, \\
H' &:= \{h \in H : h^n \in \Gamma \text{ for some } n > 0\}.
\end{aligned}$$

Then $(K', A', G') \in \mathrm{Sub}(K, A, G)$ and $(L', B', H') \in \mathrm{Sub}(L, B, H)$, and the the ordered field isomorphism $K' \cong L'$ that is identity on Δ is in $\mathcal{I}$.

To show that $\mathcal{I}$ is a back-and-forth system, let

$$\iota : (K', A', G') \to (L', B', H')$$

be in $\mathcal{I}$, and $\alpha \in K \setminus K'$. We only prove the case that $\alpha \in G$, since all the other cases distinguished in the proof of Theorem 5.3 goes through with the same proofs.

So let $\alpha \in G$. By the remark on regularly discrete pairs of ordered abelian groups, there is $\beta \in H$ realizing the cut in B' corresponding to the cut of α in A' such that for all $a' \in A'$, $g' \in G'$, $m, n > 0$ and $k, l \in \mathbb{Z}$:

$$a'\alpha^k \text{ is an } m^{\text{th}} \text{ power in } A \Longleftrightarrow \iota(a')\beta^k \text{ is an } m^{\text{th}} \text{ power in } B.$$

$$g'\alpha^l \text{ is an } n^{\text{th}} \text{ power in } G \Longleftrightarrow \iota(g')\beta^l \text{ is an } n^{\text{th}} \text{ power in } H.$$

Since B is dense in $K^{>0}$, we can choose β to realize the cut in L' corresponding to the cut of α in K'.

Then there is an isomorphism

$$\big(K'(\alpha)^{\text{rc}}, A'\langle\alpha\rangle, G'\langle\alpha\rangle\big) \to \big(L'(\beta)^{\text{rc}}, B'\langle\beta\rangle, H'\langle\beta\rangle\big),$$

which is in $\mathcal{I}$ by Lemma 5.12 of [3]. □

As in the previous case, we have the following consequence by the back-and-forth system constructed in the proof of 5.5:

Corollary 5.6. *Let $(K, A, G) \subseteq (K', A', G')$ be $\mathcal{L}_o(U, V)$-structures such that A', G' are subgroups of $(K')^{>0}$ satisfying the Mann property such that G' is a subgroup of A' with a smallest element greater than 1, and $[p]A'$ is finite for each p. Suppose that*

$$\big(K, A, G, (a')_{a' \in A'}\big) \text{ and } \big(K', A', G', (a')_{a' \in A'}\big)$$

are models of $\text{RCF}_2(A', G')$, and that A' is a pure subgroup of A. Then $(K', A', G') \preceq (K, A, G)$.

Example. Consider $\mathcal{L}_o(V)$-structures $(\mathbb{R}, 2^{\mathbb{Z}}3^{\mathbb{Z}}, 2^{\mathbb{Z}})$ and $(\mathbb{R}, 2^{\mathbb{Z}}3^{\mathbb{Z}}5^{\mathbb{Q}}, 2^{\mathbb{Z}})$. These structures satisfy the conditions of the corollary above. Thus

$$(\mathbb{R}, 2^{\mathbb{Z}}3^{\mathbb{Z}}, 2^{\mathbb{Z}}) \preceq (\mathbb{R}, 2^{\mathbb{Z}}3^{\mathbb{Z}}5^{\mathbb{Q}}, 2^{\mathbb{Z}}).$$

Now we claim that $3^{\mathbb{Z}}$ is not definable in $(\mathbb{R}, 2^{\mathbb{Z}}3^{\mathbb{Z}}, 2^{\mathbb{Z}})$. For a contradiction assume that it is definable in $(\mathbb{R}, 2^{\mathbb{Z}}3^{\mathbb{Z}}, 2^{\mathbb{Z}})$, say by the formula $\varphi(x)$. Then

$$(\mathbb{R}, 2^{\mathbb{Z}}3^{\mathbb{Z}}, 2^{\mathbb{Z}}) \models \forall x\big((V(x) \wedge \varphi(x)) \to x = 1\big),$$

$$(\mathbb{R}, 2^{\mathbb{Z}}3^{\mathbb{Z}}, 2^{\mathbb{Z}}) \models \forall x\big(U(x) \to \exists y \exists z(V(y) \wedge \varphi(z) \wedge x = yz)\big)$$

and

$$(\mathbb{R}, 2^{\mathbb{Z}}3^{\mathbb{Z}}, 2^{\mathbb{Z}}) \models \exists x\big(0 < x \wedge \varphi(x) \wedge \forall y((0 < y \wedge \varphi(y)) \to x \leq y)\big).$$

Since $2^{\mathbb{Z}}$ does not have a discrete complement in $2^{\mathbb{Z}}3^{\mathbb{Z}}5^{\mathbb{Q}}$, we get a contradiction. Hence $3^{\mathbb{Z}}$ is not definable in $(\mathbb{R}, 2^{\mathbb{Z}}3^{\mathbb{Z}}, 2^{\mathbb{Z}})$.

6. Triples of ordered abelian groups

In this section we study the triples (A, G_1, G_2), where A is an ordered abelian group, and G_1, G_2 are ordered subgroups of A such that $A = G_1 \oplus G_2$ as abelian groups. For such a triple, define $\phi : A \to A$ to be the projection of A onto G_1. Thus $\phi(A) = G_1$ and $\ker \phi = G_2$. Note that then (A, G_1, G_2) and (A, ϕ) have the same $\emptyset$-definable relations. To prove results for (A, G_1, G_2) analogous to those in Sections 3 and 4, it is convenient to study (A, ϕ) instead.

Let $\mathcal{L}_{\text{oab}}(\pi)$ be the language of ordered abelian groups augmented by a unary function symbol π. A substructure of an $\mathcal{L}_{\text{oab}}(\pi)$-structure (A, ϕ) with A' as underlying group is denoted by (A', ϕ) instead of $(A', \phi|A')$.

Let $e = \big(e(p)\big)$ and $f = \big(f(p)\big)$ be two sequences of natural numbers indexed by the prime numbers, and let $T_{\pi,e,f}$ be the $\mathcal{L}_{\text{oab}}(\pi)$-theory whose models are (A, ϕ) where

1. A is a regularly dense ordered abelian group,
2. $\phi : A \to A$ is an endomorphism of the (unordered) group A such that $\phi^2 = \phi$, and $\phi(A)$ and $\ker \phi$ are dense subgroups of A such that $A/\phi(A)$ is infinite,
3. The system of prime invariants of $(A, \phi(A))$ is e, f.

If $(A, \phi) \models T_{\pi,e,f}$, then $|\ker \phi / p \ker \phi| = p^{e(p)-f(p)}$ for all p. So $e(p) \geq f(p)$ for all p. For this reason, we assume $e(p) \geq f(p)$ for each p, in the rest of this section.

Now let (A, ϕ) and (B, ψ) be two κ-saturated models of $T_{\pi,e,f}$, where κ is an uncountable cardinal. We construct a back-and-forth system between (A, ϕ) and (B, ψ). Let $\mathcal{I}$ be the set of $\mathcal{L}_{\text{oab}}(\pi)$-isomorphisms

$$\iota : (A', \phi) \to (B', \psi),$$

where (A', ϕ) and (B', ψ) are substructures of (A, ϕ) and (B, ψ) such that

1. A' and B' are pure subgroups of A and B respectively, with $|A'|, |B'|$ are less than κ,
2. $[p]A' \geq [p]A$, and $[p]B' \geq [p]B$ for any p.

If $\iota : (A', \phi) \to (B, \psi)$ is in $\mathcal{I}$, then $\phi(A')$ and $\psi(B')$ are pure subgroups of $\phi(A)$ and $\psi(B)$ respectively. Also $[p]\phi(A') \geq [p]\phi(A)$ and $[p]\psi(B') \geq [p]\psi(B)$ for all p.

Now let $\iota : (A', \phi) \to (B, \psi)$ be in $\mathcal{I}$, and $a \in A \setminus A'$. First let $a \in \phi(A)$. Then $A'\langle a \rangle_A$ is closed under ϕ, and moreover

$$\phi(A'\langle a \rangle_A) = \phi(A')\langle a \rangle_{\phi(A)}.$$

Using the arguments of Section 3, we can find $b \in \psi(B)$, such that there is an $\mathcal{L}_{\text{oab}}(V)$-isomorphism

$$(A'\langle a \rangle_A, \phi(A')\langle a \rangle_{\phi(A)}) \to (B'\langle b \rangle_B, \psi(B')\langle b \rangle_{\psi(B)}), \quad a \mapsto b,$$

which induces an $\mathcal{L}_{\text{oab}}(\pi)$-isomorphism

$$(A'\langle a\rangle, \phi) \to (B'\langle b\rangle, \psi).$$

If $a \in A'\langle\phi(A')\rangle_A$, then we can construct an extension of ι using the previous work on pairs of regularly dense ordered abelian groups.

Now let $a \notin A'\langle\phi(A)\rangle$. Then first extend ι to an $\mathcal{L}_{\text{oab}}(\pi)$-isomorphism $(A'\langle\phi(a)\rangle, \phi) \to (B'\langle\psi(b_0)\rangle, \psi)$, with $b_0 \in B$. Now employing the arguments of Section 3 once again, take $b \in \psi^{-1}(b_0)$ such that we have an $\mathcal{L}_{\text{oab}}(V)$-isomorphism

$$(A'\langle a, \phi(A)\rangle_A, \phi) \to (B'\langle b, b_0\rangle_B, \psi),$$

which is in $\mathcal{I}$.

Corollary 6.1. *Let $(A', \phi), (A, \phi)$ be models of $T_{\pi,e,f}$ such that (A', ϕ) is a substructure of (A, ϕ) and A' is a pure subgroup of A. Then $(A', \phi) \preceq (A, \phi)$.*

Proof. Let (A^*, ϕ^*) be a κ-saturated elementary extension of (A, ϕ), where κ is an uncountable cardinal greater than $|A'|$. We may assume that (A, ϕ) is also κ-saturated. Then the identity map on A' is an element of $\mathcal{I}$. Hence we get the desired result as $\mathcal{I}$ is a back-and-forth system. □

Now enrich $\mathcal{L}_{\text{oab}}(\pi)$ by adding new distinct predicates D_n, E_n for each n, we call this new language $\mathcal{L}^*_{\text{oab}}(\pi)$. Extend $T_{\pi,e,f}$ to an $\mathcal{L}^*_{\text{oab}}(\pi)$-theory $T^*_{\pi,e,f}$ by adding the following defining axioms for D_n and E_n for each n:

$$\forall x\big(D_n(x) \leftrightarrow \exists y(x = ny)\big),$$

$$\forall x\big(E_n(x) \leftrightarrow \pi(x) = 0 \wedge \exists y((\pi(y) = 0 \wedge (x = ny)))\big).$$

By using the previous corollary and the main theorem of [7] for regularly dense ordered abelian groups, we get

Corollary 6.2. *The $\mathcal{L}^*_{\text{oab}}(\pi)$-theory $T^*_{\pi,e,f}$ is model complete.*

Now we consider the case where both $\phi(A)$ and $\ker\phi$ are regularly discrete. To do this, we augment $\mathcal{L}_{\text{oab}}(V)$ by two distinct constant symbols c_1, c_2, and call this new language $\mathcal{L}_{\text{oab2}}(V)$.

Let T_π be the $\mathcal{L}_{\text{oab2}}(\pi)$-theory whose models are the structures (A, ϕ) where

(1) A is a regularly dense ordered abelian group,
(2) $\phi : A \to A$ is an endomorphism of the (unordered) group A such that both $\phi(A)$ and $\ker\phi$ have smallest positive elements represented by the constant symbols c_1, c_2 respectively,
(3) for any $a \in A$, there are $a_1 \in A$ and $a_2 \in \ker\phi$ such that

$$\phi(a_1) \leq a < \phi(a_1) + c_1 \text{ and } a_2 \leq a < a_2 + c_2.$$

If (A, ϕ) is a model of T_π, then both $\phi(A)$ and $\ker\phi$ are regularly discrete ordered abelian groups, and thus the system of prime invariants of A is **2**.

Let (A, ϕ) and (B, ψ) be two κ-saturated models of T_π, where κ is an uncountable cardinal. Once again we construct a back-and-forth system between (A, ϕ) and (B, ψ). So let $\mathcal{I}$ be the set of all $\mathcal{L}_{\text{oab2}}(\pi)$ isomorphisms

$$\iota : (A', \phi) \to (B', \psi),$$

between $\mathcal{L}_{\text{oab2}}(\pi)$-substructures (A', ϕ) and (B', ψ) of (A, ϕ) and (B, ψ) of cardinality less than κ such that

(1) A' and B' are pure subgroups of A and B,
(2) for any $a' \in A'$, there are $a'_1 \in A')$ and $a'_2 \in \ker\phi \cap A'$ such that
$$\phi(a'_1) \leq a' < \phi(a'_1) + c_1 \text{ and } a'_2 \leq a' < a'_2 + c_2.$$
(3) for any $b' \in B'$, there are $b'_1 \in b')$ and $b'_2 \in \ker\psi \cap B'$ such that
$$\psi(b'_1) \leq b' < \psi(b'_1) + c_1 \text{ and } b'_2 \leq b' < b'_2 + c_2.$$

Now we can apply the results of the third section to extend a given element

$$\iota : (A', \phi) \to (B', \psi)$$

of $\mathcal{I}$ in a way that the extension contains a given $a \in A \setminus A'$ in its domain. Hence $\mathcal{I}$ is a back-and-forth system. We get the following result analogous to Corollary 6.1

Corollary 6.3. *Let $(A', \phi), (A, \phi)$ be models of T_π such that $(A', \phi) \subseteq (A, \phi)$ and A' is a pure subgroup of A. Then $(A', \phi) \preceq (A, \phi)$.*

References

[1] G. Birkhoff, S. MacLane, *A survey of Modern Algebra*, Third Edition, The Macmillan Co., New York; Collier-Macmillan Ltd., London 1965.
[2] L. van den Dries, The field of reals with a predicate for the powers of two, Manuscripta Math. **54** (1985), 187–195.
[3] L. van den Dries, A. Günaydin, The fields of real and complex numbers with a small multiplicative group Proceedings of London Mathematical Society, **93**, (2006), pp. 43–81.
[4] C.W. Henson, Spring 2006 Class Notes for Mathematics 571: Model Theory, available at `http://www.math.uiuc.edu/~henson/Math571/Spring2006/index.html`.
[5] C. Miller, Expansions of dense linear orders with the intermediate value property, Journal of Symbolic Logic, Vol. 66 (2001), pp. 1783–1790.
[6] Note...
[7] A. Robinson, E. Zakon, Elementary properties of ordered abelian groups, Trans. Amer. Math Soc. Vol. 96 (1960), pp. 222–236.
[8] E. Zakon, Generalized archimedean groups, Trans. Amer. Math Soc. Vol. 99 (1961), pp. 21–40.
[9] B. Zilber, A note on the model theory of the complex field with roots of unity, available at `www.maths.ox.ac.uk/~zilber`.
[10] B. Zilber, Complex roots of unity on the real plane, available at